CHAPTER 17

STATE-SPACE METHODS FOR DYNAMIC SYSTEMS ANALYSIS

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1 INTRODUCTION

The use of the state-space approach for the dynamic analysis and control of systems results in analysis and design techniques based in the time domain, as opposed to frequency-domain-based transform techniques. The state-space approach has the following characteristics:

1. It employs a more complete internal representation of dynamic systems as compared to transform methods that use input–output representations. The state of a system represents complete information about the current dynamic condition of the system. It incorporates the effect of all past inputs on the system. When combined with a complete description of the system dynamics in the form of state-space equations and knowledge of all future inputs, the future behavior of the system can be determined. More precise definitions of the notion of state are given in standard textbooks.[1–4]

2. It offers a unified approach to the analysis and synthesis of linear and nonlinear, time-invariant and time-varying, continuous-time and discrete-time, single-input and single-output, and multiple-input and multiple-output systems. Available techniques, however, are more plentiful for some categories of systems.

3. State-space-based methods rely more heavily on digital computers than classical transform-based techniques for dynamic systems analysis and control. In fact, the availability

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of digital computers both for analysis and control synthesis and for implementation of the controllers has been an important factor underlying the growing use of state-space-based methods.

4. State-space-based methods have the potential to improve the performance of controlled systems if such systems can be modeled accurately. They have been less successful in cases where system models are characterized by significant uncertainty. Classical transform-based techniques have been and continue to be widely used in such cases. In fact, one of the more encouraging trends in control systems development has been the establishment of links between state-space-based methods and transform-based methods.[5]

Though the concept of state has been invoked by a number of methods of classical mechanics and is implicit in the phase-plane concept used for nonlinear system stability analysis, the effective application of state-space-based methods for analysis and control of dynamic system behavior has occurred only over the last three decades. Pioneering theoretical work by Kalman[8–11] and others and the availability of digital computers for performing analysis and design computations have been important underlying factors. State-space methods have been most successful in aerospace control applications and less so in a variety of industrial control applications. Among the factors favoring increased emphasis in the future on state-space methods are:

1. The emphasis on controlled system performance improvement resulting from imperatives such as improved efficiency of energy utilization and improved productivity
2. The increasing availability of inexpensive but powerful digital computers for off-line analysis and design computations and online control computations

In Sections 2–7, methods for analysis of dynamic systems using state-space methods are described. Even though most of the results presented in the literature use the continuous-time formulation, the fact that digital computers will be increasingly used for controller implementation implies that discrete-time formulations have significant practical importance. Hence, both continuous-time and discrete-time formulations are presented to the fullest extent possible.

2 STATE-SPACE EQUATIONS FOR CONTINUOUS-TIME AND DISCRETE-TIME SYSTEMS

The differential equations describing the input–output behavior of an nth-order, continuous-time, nonlinear, time-varying, lumped-parameter system can be written in the form of a first-order vector ordinary differential equation and a vector output equation:

$$\mathbf{x}(t) = \mathbf{f}[\mathbf{x}(t), \mathbf{u}(t), t] \qquad t \geq t_0 \tag{1}$$

$$\mathbf{y}(t) = \mathbf{g}[\mathbf{x}(t), \mathbf{u}(t), t] \qquad t \geq t_0 \tag{2}$$

where $\mathbf{x}(t)$ = n-dimensional state vector
$\mathbf{y}(t)$ = p-dimensional output vector
$\mathbf{u}(t)$ = r-dimensional input vector
$\mathbf{f}$, $\mathbf{g}$ = vectors of appropriate dimension whose elements are single-valued nonlinear functions of the arguments noted

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Equation (1) is the state equation and Eq. (2) is the output equation. The state, output, and input vectors are

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} \qquad \mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_p(t) \end{bmatrix} \qquad \mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_r(t) \end{bmatrix} \tag{3}$$

The elements $x_1(t)$, $x_2(t)$, . . . , $x_n(t)$ of the state vector are the state variables of the system.

Formulation of the higher order system differential equations as a set of first-order differential equations has the advantage that the latter are easier to solve by numerical methods than the former. If the functions **f** and **g** are linear functions of $\mathbf{x}(t)$ and $\mathbf{u}(t)$, the system can be described by linear ordinary differential equations. Matrix notation can then be employed to simplify their representation:

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) \qquad t \geq t_0 \tag{4}$$

$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) + \mathbf{D}(t)\mathbf{u}(t) \qquad t \geq t_0 \tag{5}$$

where $\mathbf{A}(t) = n \times n$ system matrix
$\mathbf{B}(t) = n \times r$ input–state coupling matrix
$\mathbf{C}(t) = p \times n$ state–output coupling matrix
$\mathbf{D}(t) = p \times r$ input–output transmission matrix

A block diagram representation of Eqs. (4) and (5) is given in Fig. 1 using standard symbols appropriate for simulation diagrams. If the system is linear and time invariant (LTI), the matrices noted become constant matrices, as indicated by the following equations:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \qquad t \geq t_0 \tag{6}$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \qquad t \geq t_0 \tag{7}$$

If only values of the input and output variables at discrete instants in time are of interest, difference equations are appropriate for describing their relationship. The difference equations describing the input–output behavior of an nth-order, discrete-time, nonlinear, time-varying, lumped-parameter system can be written in the form of a first-order vector difference equation and a vector output equation:

$$\mathbf{x}(t_{k+1}) = \mathbf{f}[\mathbf{x}(t_k), \mathbf{u}(t_k), t_k] \qquad t_k \geq t_0 \tag{8}$$

$$\mathbf{y}(t_k) = \mathbf{g}[\mathbf{x}(t_k), \mathbf{u}(t_k), t_k] \qquad t_k \geq t_0 \tag{9}$$

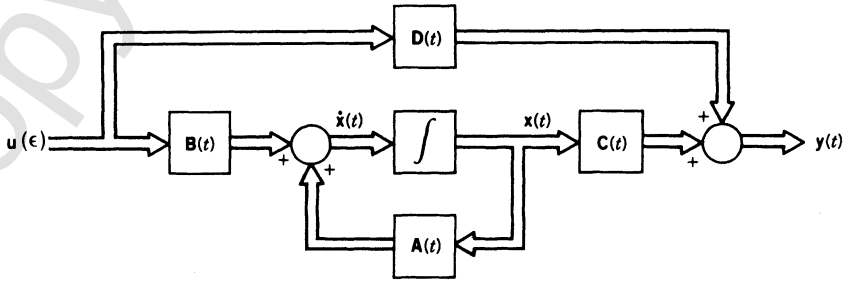


Figure 1 Linear continuous-time system.

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where $\mathbf{x}$, $\mathbf{y}$, $\mathbf{u}$ = state, output, and input vectors of the same dimensions as noted in connection with Eqs. (1) and (2)
$\mathbf{f}$, $\mathbf{g}$ = same functions as in the equations already mentioned
t_k, t_{k+1} = the kth and $(k + 1)$th discrete-time instants, respectively

If the interval between consecutive discrete-time instants is constant and equal to T, and if the functions $\mathbf{f}$ and $\mathbf{g}$ are linear, the state-space equations become

$$\mathbf{x}(k + 1) = \mathbf{F}(k)\mathbf{x}(k) + \mathbf{G}(k)\mathbf{u}(k) \qquad k \geq k_0 \tag{10}$$

$$\mathbf{y}(k) = \mathbf{C}(k)\mathbf{x}(k) + \mathbf{D}(k)\mathbf{u}(k) \qquad k \geq k_0 \tag{11}$$

where the time instants kT and $(k + 1)T$ are represented by the corresponding sequence numbers k and $k + 1$, for notational convenience. In the preceding equations, $\mathbf{F}$, $\mathbf{G}$, $\mathbf{C}$, and $\mathbf{D}$ take the place of the matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ in Eqs. (4) and (5). A block diagram representation of the system equations is given in Fig. 2. The matrices $\mathbf{F}$, $\mathbf{G}$, $\mathbf{C}$, and $\mathbf{D}$ become constant matrices for time-invariant systems:

$$\mathbf{x}(k + 1) = \mathbf{F}\mathbf{x}(k) + \mathbf{G}\mathbf{u}(k) \qquad k \geq k_0 \tag{12}$$

$$\mathbf{y}(k) = \mathbf{C}\mathbf{x}(k) + \mathbf{D}\mathbf{u}(k) \qquad k \geq k_0 \tag{13}$$

3 STATE-VARIABLE SELECTION AND CANONICAL FORMS

The state vector of a system is comprised of the minimum set of variables necessary to describe the system behavior in the form of the state-space equations already given. It can be shown that the selection of the state vector for a system is not unique.

For linear continuous-time systems, the following development shows that any vector $\mathbf{q}(t)$ related to a valid state vector selection $\mathbf{x}(t)$ by a constant nonsingular transformation matrix $\mathbf{T}$ is also a valid state vector:

$$\mathbf{q}(t) = \mathbf{T}\mathbf{x}(t) \tag{14}$$

where $\mathbf{T}$ is a nonsingular $n \times n$ matrix. Equations (4) and (5) may be rewritten in terms of the vector $\mathbf{q}(t)$ as

$$\dot{\mathbf{q}}(t) = [\mathbf{T}\mathbf{A}(t)\mathbf{T}^{-1}]\mathbf{q}(t) + [\mathbf{T}\mathbf{B}(t)]\mathbf{u}(t) \qquad t \geq t_0 \tag{15}$$

$$\mathbf{y}(t) = [\mathbf{C}(t)\mathbf{T}^{-1}]\mathbf{q}(t) + [\mathbf{D}(t)]\mathbf{u}(t) \qquad t \geq t_0 \tag{16}$$

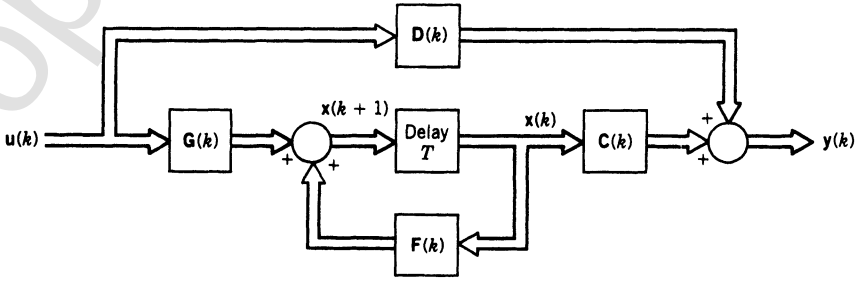


Figure 2 Linear discrete-time system.

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Here, $\mathbf{q}(t)$ satisfies the definition of a state vector since it has the same dimension as $\mathbf{x}(t)$. Equations (15) and (16) are the state-space equations in terms of $\mathbf{q}(t)$. The matrices within brackets in these equations are the modified system and coupling matrices.

As for continuous-time systems, the state vector for a given linear, discrete-time system is not unique. Any vector $\mathbf{q}(k)$ related to a valid state vector $\mathbf{x}(k)$ by a constant, nonsingular matrix $\mathbf{T}$ is also a valid state vector:

$$\mathbf{q}(k) = \mathbf{T}\mathbf{x}(k) \tag{17}$$

The corresponding state-space equations are

$$\mathbf{q}(k+1) = [\mathbf{T}\mathbf{F}(k)\mathbf{T}^{-1}]\mathbf{q}(k) + [\mathbf{T}\mathbf{G}(k)]\mathbf{u}(k) \qquad k \geq k_0 \tag{18}$$

$$\mathbf{y}(k) = [\mathbf{C}(k)\mathbf{T}^{-1}]\mathbf{q}(k) + [\mathbf{D}(k)]\mathbf{u}(k) \qquad k \geq k_0 \tag{19}$$

Since the state vector of a system is not unique, the selection of state variables for a given application is governed by considerations such as ease of measurement of state variables or simplification of the resulting state-space equations. If the independent energy storage elements in the system of interest are readily identified, selection of state variables directly related to energy storage in the system is appropriate. An nth-order system has n independent energy storage elements that would enable the selection of n state variables. Examples of energy storage elements are springs and masses in mechanical systems, capacitors and inductors in electrical systems, and capacitance and inertance elements in fluid (hydraulic and pneumatic) systems.

Consider the *RLC* circuit shown in Fig. 3. Let $e_{in}(t)$ be the input and $e_{out}(t)$ be the output. The current i_{out} is assumed to be negligible. Kirchhoff's voltage law for the loop yields

$$e_{in}(t) - Ri_{in}(t) - L\frac{di_{in}(t)}{dt} - \frac{1}{C_1}\int i_{in}(t)\,dt = 0 \tag{20}$$

The current $i_{in}(t)$ through the inductor and the voltage $e_{out}(t)$ across the capacitor are directly related to energy storage in the system and are chosen as state variables:

$$x_1(t) = i_{in}(t) \tag{21}$$

$$x_2(t) = e_{out}(t) = \frac{1}{C_1}\int i_{in}(t)\,dt \tag{22}$$

The state equations can be determined from Eqs. (20)–(22) as

$$\dot{x}_1(t) = -\frac{R}{L}x_1(t) - \frac{1}{L}x_2(t) + \frac{e_{in}(t)}{L} \tag{23}$$

$$\dot{x}_2(t) = \frac{x_1(t)}{C_1} \tag{24}$$

The output equation is

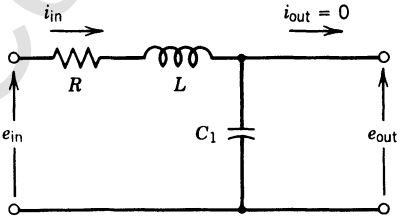


Figure 3 *RLC* circuit.

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$$y(t) = e_{\text{out}}(t) = x_2(t) \tag{25}$$

The system and coupling matrices for the electrical circuit are

$$\mathbf{A} = \begin{bmatrix} -\dfrac{R}{L} & -\dfrac{1}{L} \\ \dfrac{1}{C_1} & 0 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} \dfrac{1}{L} \\ 0 \end{bmatrix} \tag{26}$$

$$\mathbf{C} = [0 \quad 1] \qquad \mathbf{D} = 0$$

For sampled-data control applications involving digital control of continuous-time systems, state-variable selection is often based on continuous-time system equations and needs to be retained in the discrete-time formulation. Let the time-invariant, continuous-time system equations be of the form given by Eqs. (6) and (7). The solution for the vector equation of state $\mathbf{x}(t)$ is given by Eqs. (32) and (33) later in this chapter. Applying this solution form over the time interval from kT to $(k + 1)T$, we get

$$\mathbf{x}[(k + 1)T] = e^{\mathbf{A}T}\mathbf{x}(kT) + \int_{kT}^{(k+1)T} e^{\mathbf{A}[(k+1)T-t]}\mathbf{B}\mathbf{u}(t)\, dt \tag{27}$$

In sampled-data control applications, the input vector $\mathbf{u}(t)$ is the control input computed by the digital computer and applied to the continuous-time system by a digital–analog converter. Usually, the digital–analog converter has a latch that maintains the output constant between the time instants kT and $(k + 1)T$:

$$\mathbf{u}(t) = \mathbf{u}(kT) \qquad kT < t \leq (k + 1)T \tag{28}$$

Equation (27) then simplifies to

$$\mathbf{x}(k + 1) = (e^{\mathbf{A}T})\mathbf{x}(k) + \left(\int_0^T e^{\mathbf{A}(T-t)}\mathbf{B}\, dt\right)\mathbf{u}(k)$$

$$= \mathbf{F}\mathbf{x}(k) + \mathbf{G}\mathbf{u}(k) \tag{29}$$

Equation (29) establishes the relationship between the system matrices in the discrete-time and continuous-time formulations of the system equations.

3.1 Canonical Forms for Continuous-Time Systems

For high-order, single-input–single-output (SISO) systems or multiple-input–multiple-output (MIMO) systems, the number of elements in the system matrices is large. Selection of state variables that simplify the state-space representation is thus desirable. Such representations of the state-space equations also exhibit significant properties of the system more clearly and are referred to as canonical forms. However, the names and corresponding structures of the canonical forms are not completely standardized.

The controllable canonical form is useful in control system design applications. The controllable canonical form for an nth-order, SISO, LTI, continuous-time system is described in Table 1. The selection of state variables $x_1, x_2, \ldots, x_n$ that results in the controllable canonical form of the state equations is indicated on the simulation diagram in the table. For the case where all the β_i, $i = 1, n$, are zero and $\beta_0 \neq 0$, the transfer function $Y(s)/U(s)$ has no numerator dynamics. The n state variables are then simply the output and $n - 1$ successive derivatives of the output. These are referred to as the phase variables. The phase-variable canonical form is thus a special case of the controllable canonical form. The system

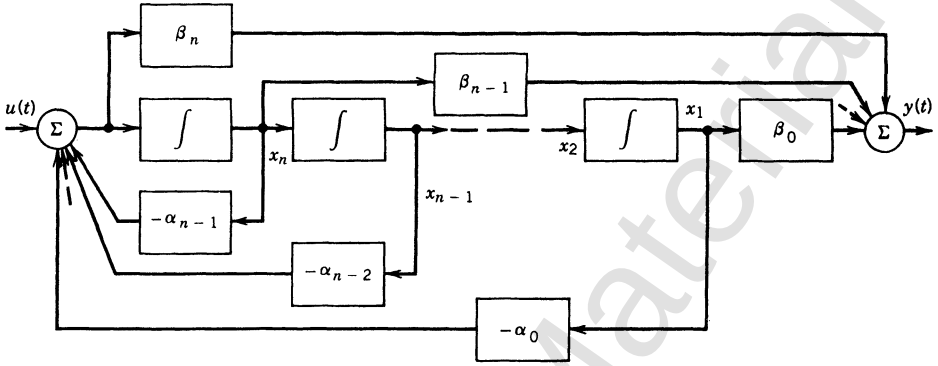
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Table 1 State-Space Canonical Forms for SISO Continuous-Time Systems

$$H(s) = \frac{Y(s)}{U(s)} = \frac{\beta_n s^n + \beta_{n-1}s^{n-1} + \cdots + \beta_1 s + \beta_0}{s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1 s + \alpha_0}$$

I. Controllable canonical form
Simulation diagram

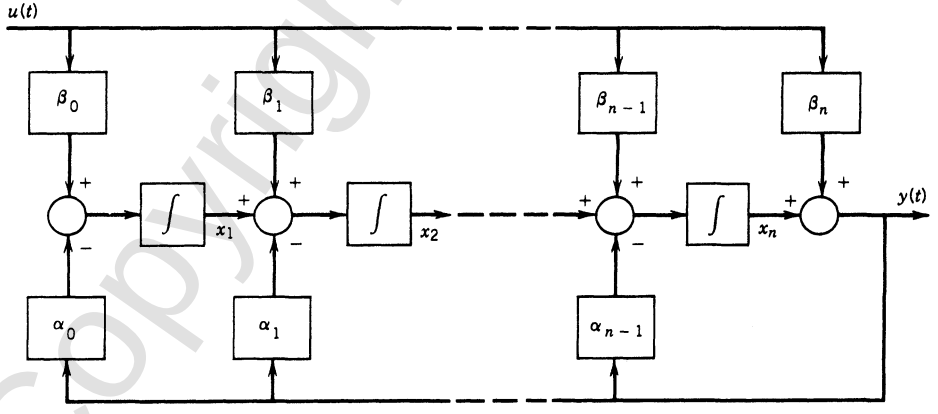


System matrices

$$\mathbf{A} = \left[\begin{array}{c|c} \begin{matrix}0\\0\\\vdots\\0\end{matrix} & \mathbf{I}_{n-1} \\ \hline -\alpha_0 & -\alpha_1 \quad \cdots \quad -\alpha_{n-1} \end{array}\right] \qquad \mathbf{B} = \begin{bmatrix}0\\0\\\vdots\\0\\1\end{bmatrix} \qquad \mathbf{D} = \beta_n$$

$$\mathbf{C} = [\beta_0 - \beta_n\alpha_0 \quad \beta_1 - \beta_n\alpha_1 \quad \ldots \quad \beta_{n-1} - \beta_n\alpha_{n-1}]$$

II. Observable canonical form
Simulation diagram



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Table 1 (*Continued*)

System matrices

$$\mathbf{A} = \left[\begin{array}{ccc:c} 0 & 0 & \cdots & -\alpha_0 \\ \hdashline & & & -\alpha_1 \\ & \mathbf{I}_{n-1} & & \vdots \\ & & & -\alpha_{n-1} \end{array}\right] \qquad \mathbf{B} = \begin{bmatrix} \beta_0 - \beta_n\alpha_0 \\ \beta_1 - \beta_n\alpha_1 \\ \vdots \\ \beta_{n-1} - \beta_n\alpha_{n-1} \end{bmatrix}$$

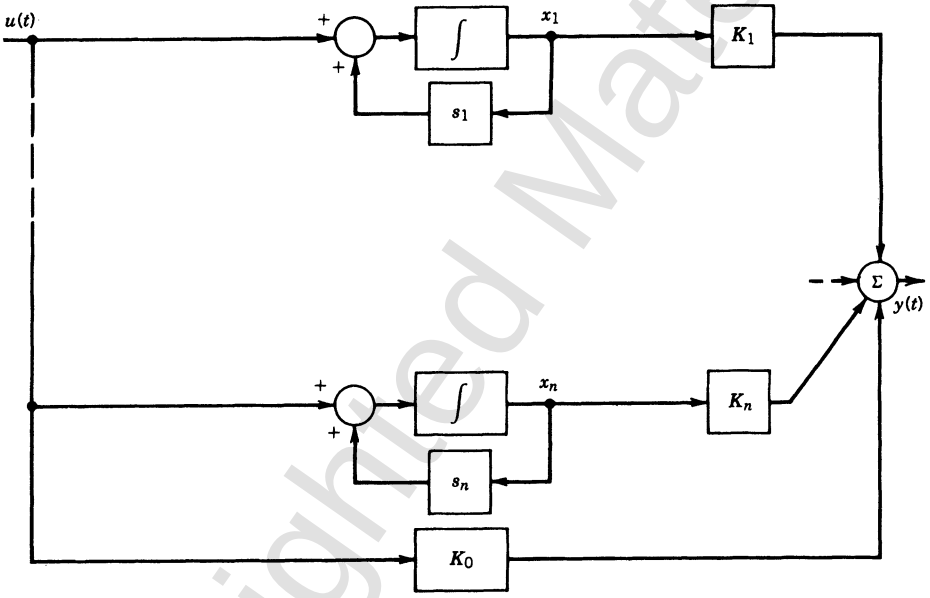
$$\mathbf{C} = [0 \quad 0 \quad \ldots \quad 0 \quad 1] \qquad \mathbf{D} = \beta_n$$

III. Normal or diagonal Jordan canonical form

Related conditions

(i) Characteristic equation roots s_i, $i = 1, \ldots, n$, are real and distinct

Simulation diagram



System matrices

$$\mathbf{A} = \begin{bmatrix} s_1 & & & \mathbf{0} \\ & s_2 & & \\ & & \ddots & \\ \mathbf{0} & & & s_n \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$\mathbf{C} = [K_1 \quad K_2 \quad \ldots \quad K_n] \qquad \mathbf{D} = K_0$$

where $K_0 = \lim_{s\to\infty} [H(s)]$ and $K_i = \lim_{s\to s_i} [(s - s_i)H(s)] \qquad i = 1, \ldots, n$

IV. Near-normal canonical form

Related conditions

(i) One pair of complex conjugate characteristic equation roots

$$s_k = s_{kr} + js_{ki}$$

$$s_{k+1} = s_{kr} - js_{ki}$$

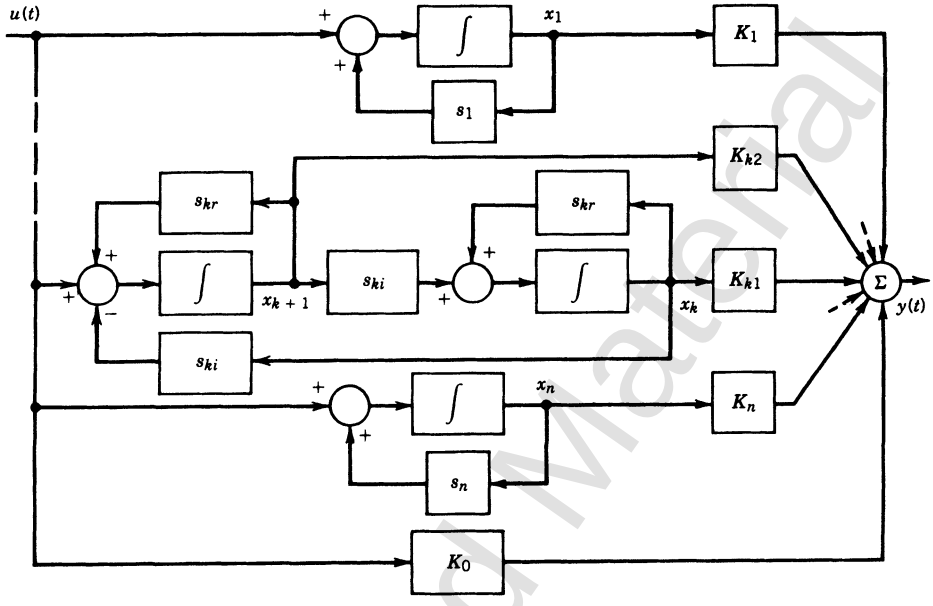
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Table 1 (*Continued*)

(ii) All other roots are real and distinct.

Simulation diagram



System matrices

$$\mathbf{A} = \left[\begin{array}{ccc|cc|cc} s_1 & & \mathbf{0} & & & \mathbf{0} & \mathbf{0} \\ & \ddots & & & & & \\ \mathbf{0} & & s_{k-1} & & & & \\ \hline & & & s_{kr} & s_{ki} & \mathbf{0} & \\ \mathbf{0} & & & -s_{ki} & s_{kr} & & \\ \hline & & & & & s_{k-2} & \mathbf{0} \\ \mathbf{0} & & \mathbf{0} & & & \ddots & \\ & & & & & \mathbf{0} & s_n \end{array}\right] \qquad \mathbf{B} = \left[\begin{array}{c} 1 \\ \vdots \\ 1 \\ \hline 0 \\ 1 \\ \hline 1 \\ \vdots \\ 1 \end{array}\right]$$

$$\mathbf{C} = [K_1 \quad \ldots \quad K_{k-1} \;|\; K_{k1} \quad K_{k2} \;|\; K_{k+2} \quad \ldots \quad K_n] \qquad \mathbf{D} = K_0$$

where $K_0 = \lim_{s\to\infty} [H(s)]$ $\qquad K_i = \lim_{s\to s_i} [(s - s_i)H(s)]$

$i = 1, k - 1$ and $i = k + 2, \ldots, n$

$$K_{k1} = -\tfrac{1}{2}\mathrm{Im}[(s - s_k)H(s)]_{s=s_k} \qquad K_{k2} = \tfrac{1}{2}\mathrm{Re}[(s - s_k)H(s)]_{s=s_k}$$

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Table 1 (*Continued*)

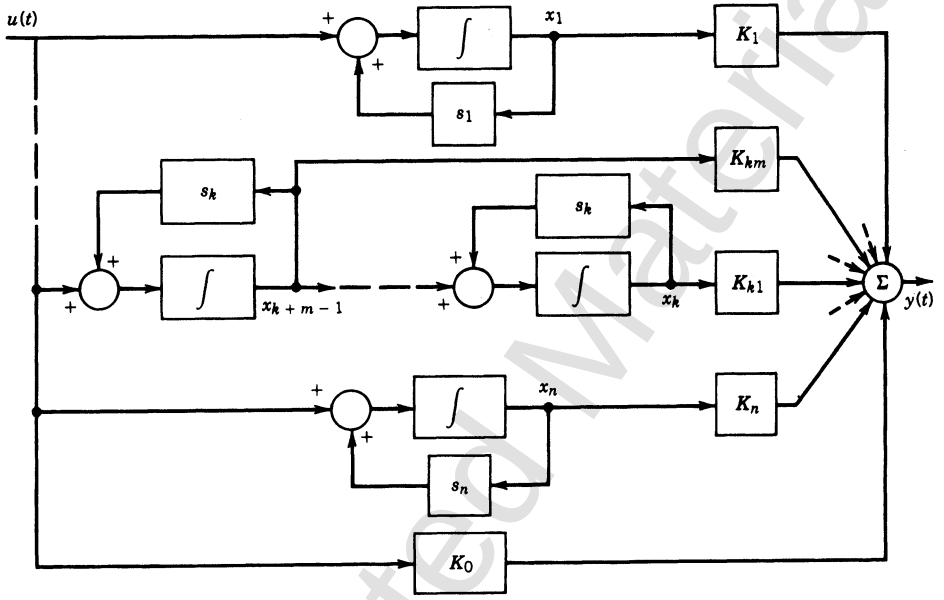
V. Nondiagonal Jordan canonical form

Related conditions

(i) One real characteristic equation root s_k repeated m times (i.e., $s_k = s_{k+1} = \cdots = s_{k+m-1}$).

(ii) All other roots are real and distinct.

Simulation diagram



System matrices

$$\mathbf{A} = \begin{bmatrix} s_1 & & 0 & & & 0 & & 0 \\ & \ddots & & & & & & \\ 0 & & s_{k-1} & & & & & \\ & & & s_k & 1 & 0 & & \\ & & & & \ddots & & & \\ 0 & & & 0 & & s_k & 0 & \\ & & & & & & s_{k+m} & 0 \\ 0 & & & 0 & & & & \ddots \\ & & & & & & 0 & & s_n \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 \\ \vdots \\ 1 \\ 0 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$\mathbf{C} = [K_1 \quad \cdots \quad K_{k-1} \mid K_{k1} \quad \cdots \quad K_{km} \mid K_{k+m} \quad \cdots \quad K_n] \qquad \mathbf{D} = K_0$$

where $K_0 = \lim_{s \to \infty} [H(s)]$ $\qquad K_i = \lim_{s \to s_i} [(s - s_i)H(s)]$

$i = 1, k - 1$ and $i = k + m, \ldots, n$

$$K_{kj} = \frac{1}{(j-1)!} \left\{ \frac{d^{j-1}}{ds^{j-1}} [(s - s_k)^m H(s)] \right\}_{s=s_k} \qquad j = 1, \ldots, m$$

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and coupling matrices for the controllable canonical form are listed in Table 1. The special form of the **A** matrix is referred to as the companion form. Here, $\mathbf{I}_{n-1}$ is the $(n-1) \times (n-1)$ identity matrix.

For a SISO, LTI system described by Eqs. (6) and (7), the state transformation matrix **T** in Eq. (14) which transforms the state-space equations into the controllable canonical form exists if the controllability matrix $\mathbf{P}_c$ in Eq. (30) is nonsingular[1]:

$$\mathbf{P}_c = [\mathbf{B} \quad \mathbf{AB} \quad \ldots \quad \mathbf{A}^{n-1}\mathbf{B}] \tag{30}$$

The transformation matrix **T** is defined in Table 2.

The observable canonical form is useful in state estimator or observer design applications. The observable canonical form for an *n*th-order, SISO, LTI system is described in Table 1 in a manner similar to the controllable canonical form. The corresponding **A** matrix is the transpose of the **A** matrix for the controllable canonical form and is also referred to as a companion matrix. The state transformation matrix **T** in Eq. (14), which transforms given state-space Eqs. (6) and (7) into the observable canonical form, exists if the observability matrix $\mathbf{P}_0$ is nonsingular[1]:

$$\mathbf{P}_0 = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix} \tag{31}$$

State variables can also be chosen to diagonalize or nearly diagonalize the state matrix **A**. The resulting state-space equations are completely or almost completely decoupled from one another and hence show very clearly the effect of initial conditions or forcing inputs on the different characteristic modes of the system response. The resulting physical insight into the system behavior makes the corresponding form of the system equations, called the normal form or diagonal Jordan form, valuable in vibration analysis applications and in control applications involving modal control. The normal form of the state-space equations for a SISO, LTI system with real, distinct characteristic roots is given in Table 1. The diagonal elements of the **A** matrix in the table are the system characteristic roots or eigenvalues. The state variables $x_1, x_2, \ldots, x_n$ lie along the eigenvectors of the **A** matrix, in state space. As the corresponding simulation diagram indicates, the behavior of each state variable is governed solely by one eigenvalue, the initial condition on that state variable, and the forcing input.

If some of the distinct characteristic roots of a SISO, LTI system are complex, the matrices **A** and **C** in Eqs. (6) and (7) have complex elements when represented in the normal form just described. Since this could be inconvenient in subsequent matrix manipulations, a nearly diagonal **A** matrix can be obtained for cases where the complex characteristic roots occur in complex-conjugate pairs. This would be the case for system differential equations with only real coefficients. The near-normal form of the system equations for a system with one pair of complex-conjugate characteristic roots is given in Table 1. Extension to the case of multiple complex root pairs is straightforward. The complex characteristic roots result in a few nonzero off-diagonal elements in the **A** matrix; otherwise, the decoupled nature of the system equations is retained.

If one characteristic root of a SISO, LTI system is real and repeated *m* times, the state equations can only be partially decoupled by appropriate state-variable selection, as shown in Table 1. The resulting state-space equations are said to be in the Jordan canonical form. The corresponding **A** matrix has one submatrix with the repeated eigenvalue at the diagonal positions, ones immediately to the right of the repeated diagonal elements within the submatrix, and zero elements at all other nondiagonal positions.[21] The **A** matrix is then said to

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Table 2 Transformation Matrices for Continuous-Time State-Space Canonical Forms

State-space equations (SISO, LTI system)
$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$
$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}u(t)$

Characteristic equation
$\det(s\mathbf{I} - \mathbf{A}) =$
$s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1 s + \alpha_0 = 0$

I. Controllable canonical form

Transformation conditions

(i) $\mathbf{P}_c = [\mathbf{B} \quad \mathbf{AB} \quad \cdots \quad \mathbf{A}^{n-1}\mathbf{B}]$ must be nonsingular

Transformation matrices

(i) $\mathbf{q} = \mathbf{T}\mathbf{x}$, $\mathbf{T} = \mathbf{R}^{-1}\mathbf{P}_c^{-1}$ where

$$\mathbf{R} = \begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} & 1 \\ \alpha_2 & \alpha_3 & \cdots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ \alpha_{n-1} & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

(ii) New state matrix $= \mathbf{T}\mathbf{A}\mathbf{T}^{-1}$

$$= \left[\begin{array}{c|c} \begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \end{matrix} & \mathbf{I}_{n-1} \\ \hline -\alpha_0 & -\alpha_1 \quad \cdots \quad -\alpha_{n-1} \end{array}\right]$$

II. Observable canonical form

Transformation conditions

(i) $\mathbf{P}_0 = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix}$ must be nonsingular

Transformation matrices

(i) $\mathbf{q} = \mathbf{T}\mathbf{x}$, $\mathbf{T} = \mathbf{R}\mathbf{P}_0$ where

$$\mathbf{R} = \begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} & 1 \\ \alpha_2 & \alpha_3 & \cdots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ \alpha_{n-1} & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

(ii) New state matrix $= \mathbf{T}\mathbf{A}\mathbf{T}^{-1}$

$$= \left[\begin{array}{ccc|c} 0 & 0 & \cdots & 0 & -\alpha_0 \\ \hline & & & & -\alpha_1 \\ & \mathbf{I}_{n-1} & & & \vdots \\ & & & & -\alpha_{n-1} \end{array}\right]$$

III. Normal or diagonal Jordan canonical form

Transformation conditions

(i) $\mathbf{A}$ matrix has only distinct, real eiganvalues s_i, $i = 1, \ldots, n$

Transformation matrices

(i) $\mathbf{q} = \mathbf{T}\mathbf{x}$, $\mathbf{T}^{-1} = \mathbf{M} = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_n]$
where (a) $\mathbf{v}_i$ are the n linearly independent eigenvectors corresponding to s_i and
(b) $\mathbf{v}_i$ are taken to be equal or proportional to any nonzero columd of $\text{Adj}(s_i\mathbf{I} - \mathbf{A})$

(ii) New state matrix $= \mathbf{T}\mathbf{A}\mathbf{T}^{-1}$

$$= \begin{bmatrix} s_1 & & & \\ & s_2 & & \mathbf{0} \\ & \mathbf{0} & \ddots & \\ & & & s_n \end{bmatrix}$$

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Table 2 (*Continued*)

IV. Normal or diagonal Jordan canonical form

Transformation conditions

(i) $\mathbf{A}$ matrix has one repeated, real eigenvalue s_k of multiplicity m (i.e., $s_k = s_{k+1} = \cdots = s_{k+m-1}$). All other eigenvalues are real and distinct.

(ii) Degeneracy $d = n - \text{rank}(s_k\mathbf{I} - \mathbf{A}) = m$. Full degeneracy.

Transformation matrices

(i) $\mathbf{q} = \mathbf{Tx}$, $\mathbf{T}^{-1} = \mathbf{M} = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_n]$

where (a) $\mathbf{v}_i$, $i = 1, k-1$ nad $i = k+m, n$, are the linearly independent eigenvectors corresponding to the real, distinct eigenvalues;

(b) $\mathbf{v}_i$, $i = 1, k-1$ and $i = k+m, n$, are taken to be equal or proportional to any nonzero column of $\text{Adj}(s_i\mathbf{I} - \mathbf{A})$; and

(c) $\mathbf{v}_i$, $i = k, k+m-1$, are the m linearly independent eigenvectors corresponding to the repeated eigenvalue. They are equal or proportional to the nonzero linearly independent columns of

$$\left\{\frac{d^{m-1}}{ds^{m-1}}\,[\text{Adj}(s\mathbf{I} - \mathbf{A})]\right\}_{s=s_k}$$

(ii) New state matrix $= \mathbf{TAT}^{-1}$

$$= \left[\begin{array}{c|c|c} \begin{matrix} s_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & s_{k-1} \end{matrix} & \begin{matrix} & \mathbf{0} & \\ s_k & & \\ & & \end{matrix} & \mathbf{0} \\ \hline \mathbf{0} & \begin{matrix} & \mathbf{0} & \\ \ddots & & s_k \\ & & \end{matrix} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \begin{matrix} s_{k+m} & & \\ & \ddots & \mathbf{0} \\ \mathbf{0} & & s_n \end{matrix} \end{array}\right]$$

V. Near-normal canonical form

Transformation conditions

(i) $\mathbf{A}$ matrix has one pair of complex-conjugate eigenvalues, s_k, s_{k+1}

$$s_k = s_{kr} + js_{ki}$$

$$s_{k+1} = s_{kr} - js_{ki}$$

(ii) All other eigenvalues are real and distinct.

Transformation matrices

(i) $\mathbf{q} = \mathbf{Tx}$, $\mathbf{T}^{-1} = [\mathbf{v}_1 \quad \cdots \quad \mathbf{v}_{k-1} \quad \mathbf{v}_{kr} \quad \mathbf{v}_{ki} \quad \mathbf{v}_{k+2} \quad \cdots \quad \mathbf{v}_n]$

where (a) $\mathbf{v}_i$, $i = 1, k-1$ and $i = k+2, n$ are the linearly independent eigenvectors corresponding to the real, distinct eigenvalues;

(b) $\mathbf{v}_i$ for $i = 1, \ldots, n$ are taken to be equal or proportional to any nonzero column of $\text{Adj}(s_i\mathbf{I} - \mathbf{A})$; and

(c) $\mathbf{v}_k = \mathbf{v}_{kr} + j\mathbf{v}_{ki}$, $\mathbf{v}_{k+1} = \mathbf{v}_{kr} - j\mathbf{v}_{ki}$

are the complex-conjugate eigenvectors corresponding to s_k and s_{k+1}, respectively.

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Table 2 (*Continued*)

(ii) New state matrix = $\mathbf{TAT}^{-1}$

$$= \left[\begin{array}{c|c|c} \begin{matrix} s_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & s_{k-1} \end{matrix} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \begin{matrix} s_{kr} & s_{ki} \\ & \ddots & \\ -s_{ki} & & s_{kr} \end{matrix} & \begin{matrix} & & \\ & \mathbf{0} & \\ & & \end{matrix} \\ \hline \mathbf{0} & \mathbf{0} & \begin{matrix} s_{k+2} & & \\ & \ddots & \mathbf{0} \\ \mathbf{0} & & s_n \end{matrix} \end{array}\right]$$

VI. Nondiagonal Jordan canonical form

Transformation conditions

(i) $\mathbf{A}$ matrix has one repeated, real eigenvalue s_k of multiplicity m (i.e., $s_k = s_{k+1} = \cdots = s_{k+m-1}$). All other eigenvalues are real and distinct.

(ii) Degeneracy $d = n - \text{rank}(s_k\mathbf{I} - \mathbf{A}) = 1$. Simple degeneracy.

Transformation matrices

(i) $\mathbf{q} = \mathbf{Tx}$, $\mathbf{T}^{-1} = [\mathbf{t}_1\ \mathbf{t}_2 \ldots \mathbf{t}_n]$

where (a) $\mathbf{t}_i$, $i = 1, k-1$ and $i = k+m, n$, are the linearly independent eigenvectors corresponding to the real, distinct eigenvalues;

(b) $\mathbf{t}_i$, for $i = 1, k-1$ and $i = k+m, n$, are taken to be equal or proportional to any nonzero column of $\text{Adj}(s_i\mathbf{I} - \mathbf{A})$; and

(c) $\mathbf{t}_i$, $i = k, k+m-1$, are obtained by solution of the equation $\mathbf{AT}^{-1} = \mathbf{T}^{-1}\mathbf{J}$, where $\mathbf{J}$ is the Jordan canonical matrix given. Each $\mathbf{t}_i$ is determined to within a constant of proportionality.

(ii) New state matrix = $\mathbf{J} = \mathbf{TAT}^{-1}$

$$= \left[\begin{array}{c|c|c} \begin{matrix} s_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & s_{k-1} \end{matrix} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \begin{matrix} s_k & 1 & \\ & \ddots & 1 \\ \mathbf{0} & & s_k \end{matrix} & \begin{matrix} & & \\ & \mathbf{0} & \\ & & \end{matrix} \\ \hline \mathbf{0} & \mathbf{0} & \begin{matrix} s_{k+m-1} & & \\ & \ddots & \mathbf{0} \\ \mathbf{0} & & s_n \end{matrix} \end{array}\right]$$

VII. Nondiagonal Jordan canonical form

Transformation conditions

(i) $\mathbf{A}$ matrix has one repeated, real eigenvalue of multiplicity m (i.e., $s_k = s_{k+1} = \cdots = s_{k+m-1}$). All other eigenvalues are real and distinct.

(ii) Degeneracy $d = n - \text{rank}(s_k\mathbf{I} - \mathbf{A})$ is between 1 and m. General degeneracy.

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Table 2 (*Continued*)

Transformation matrices

(i) $\mathbf{q} = \mathbf{Tx}$, $\mathbf{T}^{-1} = [\mathbf{t}_1\ \mathbf{t}_2\ \ldots\ \mathbf{t}_n]$

where (a) $\mathbf{t}_i$, $i = 1, k-1$ and $i = k+m, n$, are the linearly independent eigenvectors corresponding to the real, distinct eigenvalues;

(b) $\mathbf{t}_i$, $i = 1, k-1$ and $i = k+m, n$, are taken to be equal or proportional to any nonzero column of $\text{Adj}(s_i\mathbf{I} - \mathbf{A})$;

(c) $\mathbf{t}_i$, $i = k, k+m-1$, are m linearly independent vectors corresponding to the repeated eigenvalue of multiplicity m, only d of these vectors eigenvectors; and

(d) $\mathbf{t}_i$, $i = k = k, k+m-1$, obtained by solution of the equation $\mathbf{AT}^{-1} = \mathbf{T}^{-1}\mathbf{J}$, where $\mathbf{J}$ is the Jordan canonical matrix for the problem, with d Jordan blocks. There are d possible choices for $\mathbf{J}$. Each of these choices needs to be tried and the $\mathbf{t}_i$ vectors solved for. Only the correct $\mathbf{J}$ will give the m linearly independent vectors $\mathbf{t}_i$, $i = k, k+m-1$. Each $\mathbf{t}_i$ is determined to within a constant of proportionality.

(ii) New state matrix $= \mathbf{J} = \mathbf{TAT}^{-1}$. Correct $\mathbf{J}$ determined by trial and error, as previously described.

have one Jordan block. The extension of the result in Table 1 to the case of many different repeated characteristic roots is straightforward.

For SISO or MIMO, LTI systems described by state-space equations (6) and (7), the transformation matrix $\mathbf{T}$ in Eq. (14), which transforms the state-space equations into the diagonal or nondiagonal Jordan form, can be determined. As in Table 1, there are a number of different cases to be considered.[2] If the $\mathbf{A}$ matrix has real, distinct eigenvalues s_i, $i = 1, \ldots, n$, the eigenvectors are linearly independent and can be used to form the modal matrix $\mathbf{M}$ as indicated in Table 2. The transformation matrix $\mathbf{T}$ is then taken to be $\mathbf{M}^{-1}$. If the $\mathbf{A}$ matrix has one pair of complex-conjugate eigenvalues and if system matrices with real elements only are desired, the transformation matrix $\mathbf{T}$ is defined in a slightly different form as indicated in Table 2. The resulting transformed state matrix will have two nonzero off-diagonal elements as indicated.

The transformed state-space equations may be in the nondiagonal Jordan canonical form if the $\mathbf{A}$ matrix has repeated eigenvalues. The procedure for determining the transformation matrix depends on the degeneracy of the matrix $s_k\mathbf{I} - \mathbf{A}$ corresponding to the repeated eigenvalue s_k. If the degeneracy d of $s_k\mathbf{I} - \mathbf{A}$, defined in Table 2, is equal to m, where m is the multiplicity of the repeated eigenvalue s_k, m linearly independent eigenvectors can be found for the repeated eigenvalue. The procedure for doing so is indicated in Table 2. The transformed state matrix is then diagonal. If the degeneracy of $s_k\mathbf{I} - \mathbf{A}$ is 1, only one eigenvector can be determined. Since it can be shown that the degeneracy is equal to the number of Jordan blocks associated with the eigenvector, the transformed state matrix $\mathbf{J}$ has only one Jordan block and is uniquely defined.[2] The nonsingular transformation matrix $\mathbf{T}$ is then determined as indicated in Table 2. If the degeneracy d of $s_k\mathbf{I} - \mathbf{A}$ is greater than 1 but less than m, there are d linearly independent eigenvectors and d Jordan blocks associated with the eigenvalue s_k. In this case, the transformed state matrix $\mathbf{J}$ cannot be uniquely defined but can be one of a finite number of possibilities. A trial-and-error formulation of $\mathbf{J}$ and solution for $\mathbf{T}$, as indicated in Table 2, is necessary until a nonsingular transformation matrix $\mathbf{T}$ is obtained.[2]

3.2 Canonical Forms for Discrete-Time Systems

Canonical forms of discrete-time state-space equations have the same uses that such forms have for continuous-time systems. The development of these canonical forms closely par-

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allels that for continuous-time systems and is therefore summarized here. Table 3 indicates the state-variable selection for SISO, LTI systems described by pulse transfer functions that yield the controllable, observable, and Jordan canonical forms of the state-space equations. For systems already described by discrete-time state-space equations (12) and (13), Table 4 indicates the transformation matrix $\mathbf{T}$ in Eq. (17), which transforms the state-space equations into the canonical forms named previously.

With the exception of the procedures for selecting transformation matrices to convert state-space equations for MIMO, LTI systems to the Jordan canonical form, the procedures and results presented in Tables 1–4 are restricted to SISO, LTI systems. A procedure for representing a SISO, linear time-varying (LTV) system, described by a differential equation, in controllable canonical form has been described by DeRusso, Roy, and Close.[2]

Canonical forms for MIMO, LTI systems cannot, in general, be specified uniquely as is the case for SISO systems. Kailath,[3] Fortmann and Hitz,[1] and Kalman[9] have specified some canonical forms for MIMO systems and have described procedures for representing such systems in these forms, given their transfer function matrix descriptions or state-space equations. The problem of selection of state variables given the transfer function matrix description of MIMO systems is the problem of realization and is considered in Section 7.

4 SOLUTION OF SYSTEM EQUATIONS

4.1 Continuous-Time Systems

The state-space Eqs. (1) and (2) for time-varying, nonlinear systems described by ordinary differential equations can be solved by numerical integration techniques. Such numerical integration would, however, have to be repeated if the initial conditions $\mathbf{x}(t_0)$ or the forcing function $\mathbf{u}(t)$ were to be changed. The computational burden can be reduced for linear systems by using the concept of the state transition matrix.

For LTI systems described by the state equations (6) and (7), the solution is given by

$$\mathbf{x}(t) = \boldsymbol{\phi}(t - t_0)\mathbf{x}(t_0) + \int_{t_0}^{t} \boldsymbol{\phi}(t - \tau)\mathbf{B}\mathbf{u}(\tau)\, d\tau \tag{32}$$

$$\mathbf{y}(t) = \mathbf{C}\boldsymbol{\phi}(t - t_0)\mathbf{x}(t_0) + \int_{t_0}^{t} \mathbf{C}\boldsymbol{\phi}(t - \tau)\mathbf{B}\mathbf{u}(\tau)\, d\tau + \mathbf{D}\mathbf{u}(t) \tag{33}$$

where the $n \times n$ matrix $\boldsymbol{\phi}(t)$ is defined as the state transition matrix of the system. Derivation of this result is available in many standard textbooks on state-space methods.[1,4] The first terms on the right-hand sides of the preceding equations represent the response of the homogeneous system to the initial condition $\mathbf{x}(t_0)$ whereas the second terms represent the forced response of the system. Comparison of the second term in Eq. (33), for the case $\mathbf{D} = 0$, with Eq. (34) for the forced response of a SISO, LTI system indicates that the matrix $\mathbf{C}\boldsymbol{\phi}(t)\mathbf{B}$ is a matrix of impulse responses:

$$y(t_0) = y(t_0) + \int_{t_0}^{t} h(t - \tau)u(\tau)\, d\tau \tag{34}$$

The variable $h(t)$ in Eq. (34) is the impulse response of the system. The interpretation of $\mathbf{C}\boldsymbol{\phi}(t)\mathbf{B}$ as a matrix of impulse responses forms the basis of one of the methods for determining the elements of the transition matrix.[2,4]

The state transition matrix $\boldsymbol{\phi}(t - t_0)$ is the solution of the matrix differential equation

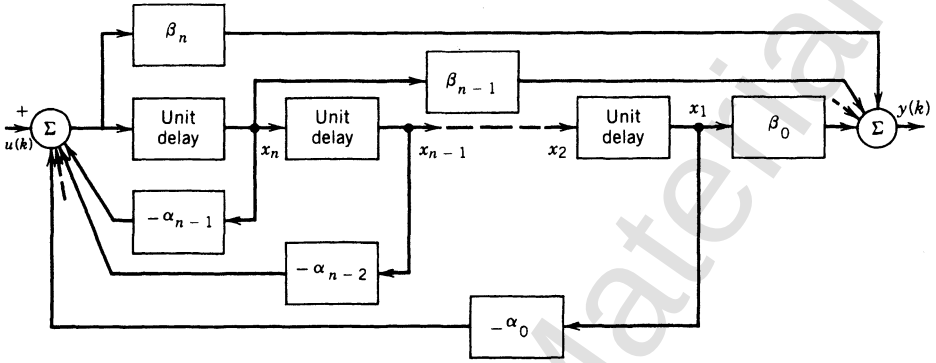
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Table 3 State-Space Canonical Forms for SISO Discrete-Time Systems

$$H(z) = \frac{Y(z)}{U(z)} = \frac{\beta_n z^n + \beta_{n-1} z^{n-1} + \cdots + \beta_1 z + \beta_0}{z^n + \alpha_{n-1} z^{n-1} + \cdots + \alpha_1 z + \alpha_0}$$

I. Controllable canonical form
Simulation diagram

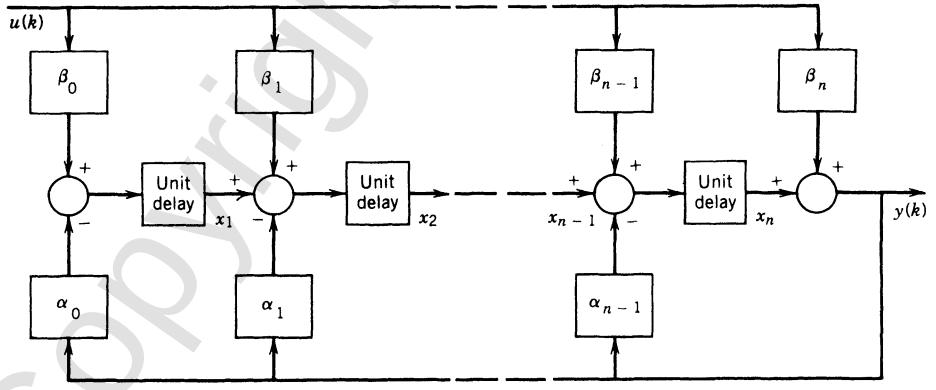


System matrices

$$\mathbf{F} = \left[\begin{array}{c|ccc} 0 & & & \\ 0 & & & \\ \vdots & & \mathbf{I}_{n-1} & \\ 0 & & & \\ \hline -\alpha_0 & -\alpha_1 & \cdots & -\alpha_{n-1} \end{array}\right] \qquad \mathbf{G} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{D} = \beta_n$$

$$\mathbf{C} = [\beta_0 - \beta_n\alpha_0 \quad \beta_1 - \beta_n\alpha_1 \quad \ldots \quad \beta_{n-1} - \beta_n\alpha_{n-1}]$$

II. Observable canonical form
Simulation diagram



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Table 3 (*Continued*)

System matrices

$$\mathbf{F} = \left[\begin{array}{ccc:c} 0 & 0 & \cdots & -\alpha_0 \\ \hdashline & & & -\alpha_1 \\ & \mathbf{I}_{n-1} & & \vdots \\ & & & -\alpha_{n-1} \end{array}\right] \qquad \mathbf{G} = \begin{bmatrix} \beta_0 - \beta_n\alpha_0 \\ \beta_1 - \beta_n\alpha_1 \\ \vdots \\ \beta_{n-1} - \beta_n\alpha_{n-1} \end{bmatrix}$$

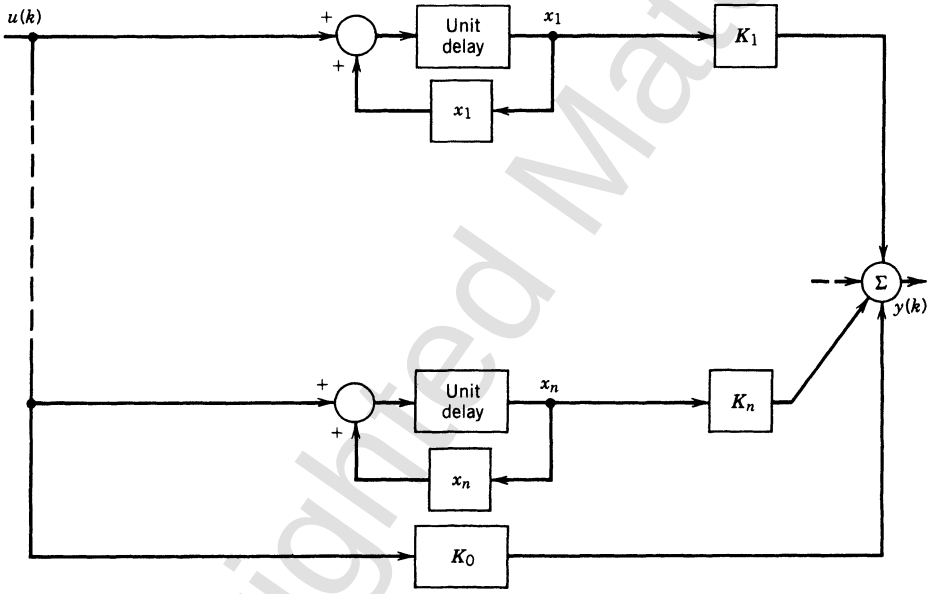
$$\mathbf{C} = [0 \quad 0 \quad \cdots \quad 0 \; 1] \qquad \mathbf{D} = \beta_n$$

III. Normal or diagonal Jordan canonical form

Related conditions

(i) Characteristic equation roots z_i, $i = 1, \ldots, n$, are real and distinct.

Simulation diagram



System matrices

$$\mathbf{F} = \begin{bmatrix} z_1 & & & \mathbf{0} \\ & z_2 & & \\ & & \ddots & \\ \mathbf{0} & & & z_n \end{bmatrix} \qquad \mathbf{G} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$\mathbf{C} = [K_1 \quad K_2 \quad \cdots \quad K_n] \qquad \mathbf{D} = K_0$$

where $K_0 = \lim_{z \to \infty} [H(z)]$ and $K_i = \lim_{z \to z_i} [(z - z_i)H(z)] \qquad i = 1, \ldots, n$

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Table 3 (*Continued*)

IV. Near-normal canonical form

Related conditions

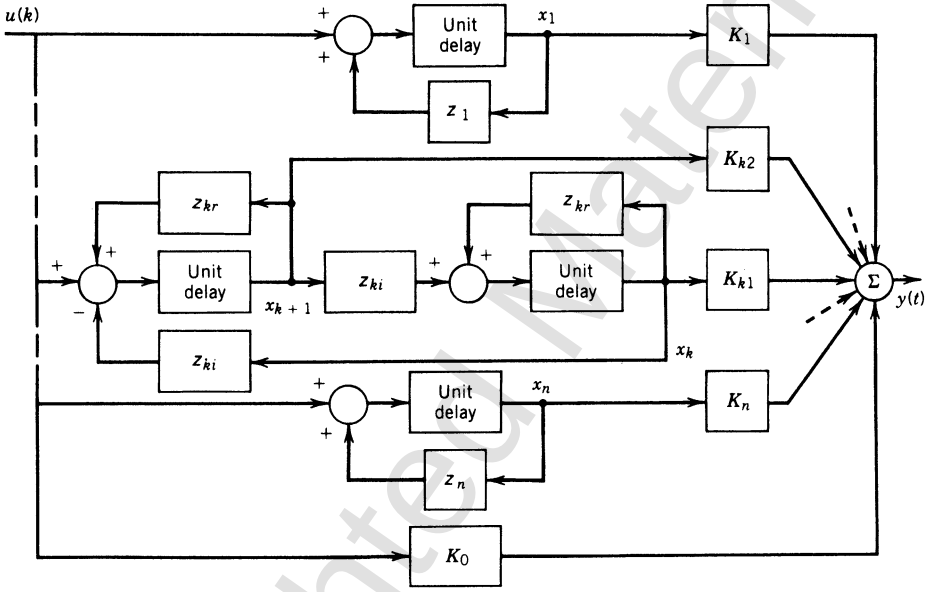
(i) One pair of complex conjugate characteristic equation roots

$$z_k = z_{kr} + jz_{ki}$$

$$z_{k+1} = z_{kr} - jz_{ki}$$

(ii) All other roots are real and distinct.

Simulation diagram



System matrices

$$\mathbf{F} = \left[\begin{array}{c|c|c} \begin{matrix} z_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & z_{k-1} \end{matrix} & \begin{matrix} & \mathbf{0} & \\ z_{kr} & & z_{ki} \\ -z_{ki} & & z_{kr} \end{matrix} & \begin{matrix} \mathbf{0} \\ \mathbf{0} \end{matrix} \\ \hline \mathbf{0} & & \mathbf{0} \\ \hline \begin{matrix} \mathbf{0} & & \mathbf{0} \end{matrix} & \begin{matrix} & \mathbf{0} & \\ z_{k-2} & & \mathbf{0} \\ \mathbf{0} & & z_n \end{matrix} & \end{array}\right] \qquad \mathbf{G} = \begin{bmatrix} 1 \\ \vdots \\ 1 \\ \hline 0 \\ 1 \\ \hline 1 \\ \vdots \\ 1 \end{bmatrix}$$

where $K_0 = \lim_{z\to\infty} [H(z)]$ $\qquad K_i = \lim_{z\to z_i} [(z - z_i)H(z)]$

$i = 1, k - 1$ and $i = k + 2, \ldots, n$

$K_{k1} = -\frac{1}{2}\text{Im}[(z - z_k)H(z)]_{z=z_k}$ $\qquad K_{k2} = \frac{1}{2}\text{Re}[(z - z_k)H(z)]_{z=z_k}$

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Table 3 (*Continued*)

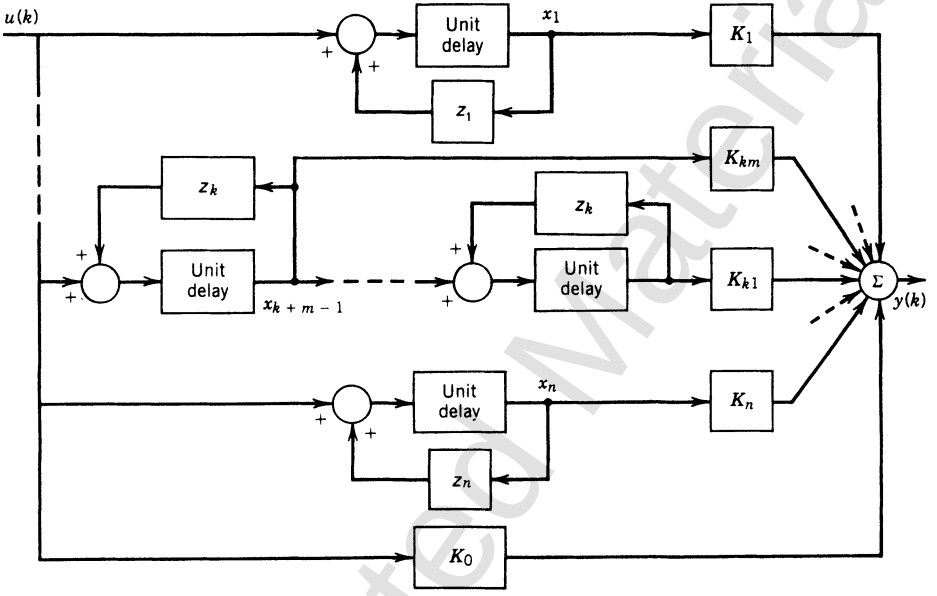
V. Nondiagonal Jordan canonical form

Related conditions

(i) One real characteristic equation root z_k repeated m times (i.e., $z_k = z_{k+1} = \cdots = z_{k+m-1}$).

(ii) All other roots are real and distinct.

Simulation diagram



System matrices

$$\mathbf{F} = \left[\begin{array}{ccc|c|c} z_1 & & \mathbf{0} & & \\ & \ddots & & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & & z_{k-1} & & \\ \hline & & & z_k \; 1 & \\ \mathbf{0} & & & \mathbf{0} \quad \ddots & \mathbf{0} \\ & & & \mathbf{0} \quad 1 & \\ & & & z_k & \\ \hline & & & & z_{k+m} \; \mathbf{0} \\ \mathbf{0} & & \mathbf{0} & & \ddots \\ & & & \mathbf{0} & z_n \end{array}\right] \qquad \mathbf{G} = \left[\begin{array}{c} 1 \\ \vdots \\ 1 \\ \hline 0 \\ 1 \\ \hline 1 \\ \vdots \\ 1 \end{array}\right]$$

$$\mathbf{C} = [K_1 \quad \cdots \quad K_{k-1} \;\vdots\; K_{k1} \quad \cdots \quad K_{km} \;\vdots\; K_{k+m} \quad \cdots \quad K_n] \qquad \mathbf{D} = K_0$$

where $K_0 = \lim_{z \to \infty} [H(z)]$ $\qquad K_i = \lim_{z \to z_i} [(z - z_i)H(z)]$

$i = 1, k - 1$ and $i = k + m, \ldots, n$

and $K_{kj} = -\dfrac{1}{(j-1)!} \left\{ \dfrac{d^{j-1}}{js^{j-1}} [(z - z_k)^m H(z)] \right\}_{z=z_k}$ $\qquad j = 1, \ldots, m$

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Table 4 Transformation Matrices for Discrete-Time State-Space Canonical Forms

State-space equations (SISO system)
$\mathbf{x}(k+1) = \mathbf{F}\mathbf{x}(k) + \mathbf{G}u(k)$
$y(k) = \mathbf{C}\mathbf{x}(k) + \mathbf{D}u(k)$

Characteristic equation
$\det(z\mathbf{I} - \mathbf{F}) = z^n + \alpha_{n-1}z^{n-1} + \cdots + \alpha_1 z + \alpha_0 = 0$

I. Controllable canonical form

Transformation conditions

(i) $\mathbf{P}_c = [\mathbf{G} \quad \mathbf{FG} \quad \cdots \quad \mathbf{F}^{n-1}\mathbf{G}]$ must be nonsingular

Transformation matrices

(i) $\mathbf{q} = \mathbf{T}\mathbf{x}$, $\mathbf{T} = \mathbf{R}^{-1}\mathbf{P}_c^{-1}$ where

$$\mathbf{R} = \begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} & 1 \\ \alpha_2 & \alpha_3 & \cdots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ \alpha_{n-1} & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

(ii) New state matrix $= \mathbf{TFT}^{-1}$

$$= \left[\begin{array}{c:ccc} 0 & & & \\ 0 & & & \\ \vdots & & \mathbf{I}_{n-1} & \\ 0 & & & \\ \hdashline -\alpha_0 & -\alpha_1 & \cdots & -\alpha_{n-1} \end{array}\right]$$

II. Observable canonical form

Transformation conditions

(i) $\mathbf{P}_0 = \begin{bmatrix} \mathbf{C} \\ \mathbf{CF} \\ \vdots \\ \mathbf{CF}^{n-1} \end{bmatrix}$ must be nonsingular

Transformation matrices

(i) $\mathbf{q} = \mathbf{T}\mathbf{x}$, $\mathbf{T} = \mathbf{R}\mathbf{P}_0$ where

$$\mathbf{R} = \begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} & 1 \\ \alpha_2 & \alpha_3 & \cdots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ \alpha_{n-1} & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

(ii) New state matrix $= \mathbf{TFT}^{-1}$

$$= \left[\begin{array}{ccc:c} 0 & 0 & \cdots & 0 & -\alpha_0 \\ & & & & -\alpha_1 \\ & & \mathbf{I}_{n-1} & & \vdots \\ & & & & -\alpha_{n-1} \end{array}\right]$$

III. Normal or diagonal Jordan canonical form

Transformation conditions

(i) $\mathbf{F}$ matrix has only distinct, real eiganvalues z_i, $i = 1, \ldots, n$

Transformation matrices

(i) $\mathbf{q} = \mathbf{T}\mathbf{x}$, $\mathbf{T}^{-1} = \mathbf{M} = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_n]$
where (a) $\mathbf{v}_i$ are the n linearly independent eigenvectors corresponding to z_i and
(b) $\mathbf{v}_i$ are taken to be equal or proportional to any nonzero column of $\text{Adj}(z_i\mathbf{I} - \mathbf{F})$

(ii) New state matrix $= \mathbf{TFT}^{-1}$

$$= \begin{bmatrix} z_1 & & & \mathbf{0} \\ & z_2 & & \\ & & \ddots & \\ \mathbf{0} & & & z_n \end{bmatrix}$$

IV. Normal or diagonal Jordan canonical form

Transformation conditions

(i) $\mathbf{F}$ matrix has one repeated, real eigenvalue z_k of multiplicity m (i.e., $z_k = z_{k+1} = \ldots = z_{k+m-1}$). All other eigenvalues are real and distinct.

(ii) Degeneracy $d = n - \text{rank}(z_k\mathbf{I} - \mathbf{F}) = m$. Full degeneracy.

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Table 4 (*Continued*)

Transformation matrices

(i) $\mathbf{q} = \mathbf{Tx}$, $\mathbf{T}^{-1} = \mathbf{M} = [\mathbf{v}_1\ \mathbf{v}_2\ \ldots\ \mathbf{v}_n]$

where (a) $\mathbf{v}_i$, $i = 1, k-1$ nad $i = k+m, n$, are the linearly independent eigenvectors corresponding to the real, distinct eigenvalues;

(b) $\mathbf{v}_i$, $i = 1, k-1$ and $i = k+m, n$, are taken to be equal or proportional to any nonzero column of $\text{Adj}(z_i\mathbf{I} - \mathbf{F})$; and

(c) $\mathbf{v}_i$, $i = k, k+m-1$, are the m linearly independent eigenvectors corresponding to the repeated eigenvalue. They are equal or proportional to the nonzero linearly independent columns of

$$\left\{\frac{d^{m-1}}{dz^{m-1}}\,[\text{Adj}(s\mathbf{I} - \mathbf{F})]\right\}_{z=z_k}$$

(ii) New state matrix $= \mathbf{TFT}^{-1}$

$$= \left[\begin{array}{ccc|c|c} z_1 & & & & \\ & \ddots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & & z_{k-1} & & \\ \hline & & & z_k & \\ \mathbf{0} & & & \ddots\ \mathbf{0} & \mathbf{0} \\ & & & \mathbf{0}\quad z_k & \\ \hline & & & & z_{k+m} \\ \mathbf{0} & & \mathbf{0} & & \ddots\ \mathbf{0} \\ & & & \mathbf{0} & z_n \end{array}\right]$$

V. Near-normal canonical form

Transformation conditions

(i) $\mathbf{F}$ matrix has one pair of complex-conjugate eigenvalues, z_k, z_{k+1}

$$z_k = z_{kr} + jz_{ki}$$

$$z_{k+1} = z_{kr} - jz_{ki}$$

(ii) All other eigenvalues are real and distinct.

Transformation matrices

(i) $\mathbf{q} = \mathbf{Tx}$, $\mathbf{T}^{-1} = [\mathbf{v}_1\ \cdots\ \mathbf{v}_{k-1}\ \ \mathbf{v}_{kr}\ \ \mathbf{v}_{ki}\ \ \mathbf{v}_{k+2}\ \ \cdots\ \ \mathbf{v}_n]$

where (a) $\mathbf{v}_i$, $i = 1, k-1$ and $i = k+2, \ldots, n$, are the linearly independent eigenvectors corresponding to the real, distinct eigenvalues;

(b) $\mathbf{v}_i$ for $i = 1, \ldots, n$ are taken to be equal or proportional to any nonzero column of $\text{Adj}(z_i\mathbf{I} - \mathbf{F})$; and

(c) $\mathbf{v}_k = \mathbf{v}_{kr} + j\mathbf{v}_{ki}$, $\mathbf{v}_{k+1} = \mathbf{v}_{kr} - j\mathbf{v}_{ki}$

are the complex-conjugate eigenvectors corresponding to z_k and z_{k+1}, respectively.

(ii) New state matrix $= \mathbf{TFT}^{-1}$

$$= \left[\begin{array}{ccc|c|c} z_1 & & & & \\ & \ddots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & & z_{k-1} & & \\ \hline & & & z_{kr}\quad z_{ki} & \\ \mathbf{0} & & & \ddots & \mathbf{0} \\ & & & -z_{ki}\quad z_{kr} & \\ \hline & & & & z_{k+2} \\ \mathbf{0} & & \mathbf{0} & & \ddots\ \mathbf{0} \\ & & & \mathbf{0} & z_n \end{array}\right]$$

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Table 4 (*Continued*)

VI. Nondiagonal Jordan canonical form

Transformation conditions

(i) $\mathbf{F}$ matrix has one repeated, real eigenvalue z_k of multiplicity m (i.e., $z_k = z_{k+1} = \cdots = z_{k+m-1}$). All other eigenvalues are real and distinct.

(ii) Degeneracy $d = n - \text{rank}(z_k\mathbf{I} - \mathbf{F}) = 1$. Simple degeneracy.

Transformation matrices

(i) $\mathbf{q} = \mathbf{Tx}$, $\mathbf{T}^{-1} = [\mathbf{t}_1\ \mathbf{t}_2 \ldots \mathbf{t}_n]$

where (a) $\mathbf{t}_i$, $i = 1, k-1$ and $i = k+m, n$, are the linearly independent eigenvectors corresponding to the real, distinct eigenvalues;

(b) $\mathbf{t}_i$, for $i = 1, k-1$ and $i = k+m, n$, are taken to be equal or proportional to any nonzero column of $\text{Adj}(s_i\mathbf{I} - \mathbf{F})$; and

(c) $\mathbf{t}_i$, $i = k, k+m-1$, are obtained by solution of the equation $\mathbf{FT}^{-1} = \mathbf{T}^{-1}\mathbf{J}$, where $\mathbf{J}$ is the Jordan canonical matrix given. Each $\mathbf{t}_i$ is determined to within a constant of proportionality.

(ii) New state matrix $= \mathbf{J} = \mathbf{TFT}^{-1}$

$$= \left[\begin{array}{c|c|c} \begin{matrix} z_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & z_{k-1} \end{matrix} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \begin{matrix} z_k & 1 & & \mathbf{0} \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ \mathbf{0} & & & z_k \end{matrix} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \begin{matrix} z_{k+m-1} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & z_n \end{matrix} \end{array}\right]$$

VII. Nondiagonal Jordan canonical form

Transformation conditions

(i) $\mathbf{F}$ matrix has one repeated, real eigenvalue of multiplicity m (i.e., $z_k = z_{k+1} = \cdots = z_{k+m-1}$). All other eigenvalues are real and distinct.

(ii) Degeneracy $d = n - \text{rank}(z_k\mathbf{I} - \mathbf{F})$ is between 1 and m. Simple degeneracy.

Transformation matrices

(i) $\mathbf{q} = \mathbf{Tx}$, $\mathbf{T}^{-1} = [\mathbf{t}_1\ \ \mathbf{t}_2\ \ \ldots\ \ \mathbf{t}_n]$

where (a) $\mathbf{t}_i$, $i = 1, k-1$ and $i = k+m, n$, are the linearly independent eigenvectors corresponding to the real, distinct eigenvalues;

(b) $\mathbf{t}_i$, $i = 1, k-1$ and $i = k+m, n$, are taken to be equal or proportional to any nonzero column of $\text{Adj}(s_i\mathbf{I} - \mathbf{F})$;

(c) $\mathbf{t}_i$, $i = k, k+m-1$, are m linearly independent vectors corresponding to the repeated eigenvalue of multiplicity m, only d of these vectors eigenvectors; and

(d) $\mathbf{t}_i$, $i = k = k, k+m-1$, obtained by solution of the equation $\mathbf{FT}^{-1} = \mathbf{T}^{-1}\mathbf{J}$, where $\mathbf{J}$ is the Jordan canonical matrix for the problem, with d Jordan blocks. There are d possible choices for $\mathbf{J}$. Each of these choices needs to be tried and the $\mathbf{t}_i$ vectors solved for. Only the correct $\mathbf{J}$ will give the m linearly independent vectors $\mathbf{t}_i$, $i = k, k+m-1$. Each $\mathbf{t}_i$ is determined to within a constant of proportionality.

(ii) New state matrix $= \mathbf{J} = \mathbf{TFT}^{-1}$. Correct $\mathbf{J}$ determined by trial and error, as previously described.

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$$\boldsymbol{\phi}(t - t_0) = \mathbf{A}\boldsymbol{\phi}(t - t_0) \qquad t \geq t_0 \tag{35}$$

with the initial condition

$$\boldsymbol{\phi}(t_0 - t_0) = \boldsymbol{\phi}(0) = \mathbf{I} \tag{36}$$

It has the following properties:

$$\boldsymbol{\phi}(t + \tau) = \boldsymbol{\phi}(t)\boldsymbol{\phi}(\tau) = \boldsymbol{\phi}(\tau)\boldsymbol{\phi}(t) \tag{37}$$

$$\boldsymbol{\phi}^{-1}(t) = \boldsymbol{\phi}(-t) \tag{38}$$

The following expressions for $\boldsymbol{\phi}(t)$ can be verified and are useful in its evaluation:

$$\boldsymbol{\phi}(t) = e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2t^2}{2!} + \frac{\mathbf{A}^3t^3}{3!} + \cdots \tag{39}$$

and

$$\boldsymbol{\phi}(t) = \mathcal{L}^{-1}[(s\mathbf{I} - \mathbf{A})^{-1}] \tag{40}$$

where $\mathcal{L}^{-1}$ denotes the inverse Laplace transform. Details related to Eqs. (35)–(40) have been described by DeRusso et al.[2] and Brogan.[4]

Knowledge of the state transition matrix for a given system simplifies the task of determining the response of the system to a variety of initial conditions $\mathbf{x}(t_0)$ and forcing functions $\mathbf{u}(t)$. A number of analytical and numerical techniques for its evaluation are available.

Equation (39) forms the basis for a numerical method of determining $\boldsymbol{\phi}(t)$. Closed-form evaluation of $e^{\mathbf{A}t}$ is possible only for special forms of the $\mathbf{A}$ matrix. For example, if $\mathbf{A}$ is a diagonal matrix with diagonal elements equal to the eigenvalues s_i, it can be shown that $\boldsymbol{\phi}(t)$ is also diagonal[4] and is given by

$$\boldsymbol{\phi}(t) = e^{\mathbf{A}t} = \begin{bmatrix} e^{s_1t} & & & \\ & e^{s_2t} & & \mathbf{0} \\ & & \ddots & \\ \mathbf{0} & & & e^{s_nt} \end{bmatrix} \tag{41}$$

Closed-form evaluation of $e^{\mathbf{A}t}$ is only slightly more complex if $\mathbf{A}$ is in the nondiagonal Jordan canonical form.[4] If the transformation matrix $\mathbf{T}$ [Eq. (14)] was used to obtain the diagonal or nondiagonal Jordan matrix $\mathbf{A}$, the transition matrix, for the original state vector $\mathbf{T}^{-1}\mathbf{x}$, is $\mathbf{T}^{-1}e^{\mathbf{A}t}\mathbf{T}$.

Equation (40) provides the basis for an analytical evaluation of $\boldsymbol{\phi}(t)$ that is suitable for low-order dynamic systems. This method requires the inversion of the $n \times n$ matrix $s\mathbf{I} - \mathbf{A}$, followed by the inverse Laplace transformation of the n^2 elements. The matrix inversion is especially cumbersome since the elements of the matrix are functions of s. The matrix inversion can be avoided altogether by using simulation diagrams of the system, in conjunction with block diagram reduction techniques,[2] to determine elements of the matrix $(s\mathbf{I} - \mathbf{A})^{-1}$. Alternative analytical techniques for the evaluation of $\boldsymbol{\phi}(t)$ based on Sylvester's theorem and the Cayley–Hamilton theorem have been described by DeRusso et al.[2] and Brogan.[4]

Numerical evaluation of $\boldsymbol{\phi}(t)$ for a specified value of t can be performed using Eq. (39) and retaining a finite number of terms from the series expansion. The number of terms retained increases with the desired degree of accuracy. An iterative procedure for determining the number of terms to be retained for a specified degree of accuracy has been described by Shinners.[5]

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For linear, time-varying systems described by state-space equations (4) and (5), the solution is given by[2,4]

$$\mathbf{x}(t) = \boldsymbol{\phi}(t, t_0)\mathbf{x}(t_0) + \int_{t_0}^{t} \boldsymbol{\phi}(t, \tau)\mathbf{B}(\tau)\mathbf{u}(\tau)\, d\tau \qquad t \geq t_0 \tag{42}$$

$$\mathbf{y}(t) = \mathbf{C}(t)\boldsymbol{\phi}(t, t_0)\mathbf{x}(t_0) + \int_{t_0}^{t} \mathbf{C}(t)\boldsymbol{\phi}(t, \tau)\mathbf{B}(\tau)\mathbf{u}(\tau)\, d\tau + \mathbf{D}(t)\mathbf{u}(t) \tag{43}$$

where the $n \times n$ state transition matrix $\boldsymbol{\phi}(t, t_0)$ for the time-varying system depends on both arguments t and t_0 and not merely on the difference between these two time instants as in the time-invariant system.

The state transition matrix $\boldsymbol{\phi}(t, t_0)$ is the solution of the partial differential equation[2,4]

$$\frac{\partial \boldsymbol{\phi}(t, t_0)}{\partial t} = \mathbf{A}(t)\boldsymbol{\phi}(t, t_0) \tag{44}$$

with the initial condition

$$\boldsymbol{\phi}(t_0, t_0) = \mathbf{I} \tag{45}$$

It has the following properties:

$$\boldsymbol{\phi}(t_2, t_0) = \boldsymbol{\phi}(t_2, t_1)\boldsymbol{\phi}(t_1, t_0) \tag{46}$$

$$\boldsymbol{\phi}(t_1, t_0) = \boldsymbol{\phi}^{-1}(t_0, t_1) \tag{47}$$

Techniques for evaluating the state transition matrix for time-varying systems are considerably more involved than for time-invariant systems and are less widely applicable. A number of analytical methods for determining $\boldsymbol{\phi}(t, t_0)$ for special cases of linear time-varying systems have been described by DeRusso et al.[2]

A simple numerical procedure has been suggested by Palm[6] for computing the transition matrix when analytical determination is not possible. Let the ith column of $\boldsymbol{\phi}(t, t_0)$ be denoted by $\boldsymbol{\psi}_i(t)$ for a specified value of t_0. The matrix partial differential equation (44) becomes n vector ordinary differential equations:

$$\dot{\boldsymbol{\psi}}_i(t) = \mathbf{A}(t)\boldsymbol{\psi}_i(t) \qquad i = 1, \ldots, n \qquad t \geq t_0 \tag{48}$$

with the initial conditions

$$[\boldsymbol{\psi}_1(t_0) \quad \boldsymbol{\psi}_2(t_0) \quad \cdots \quad \boldsymbol{\psi}_n(t_0)] = \mathbf{I} \tag{49}$$

Numerical solution of the ordinary differential equations gives $\boldsymbol{\psi}_i(t)$ and hence $\boldsymbol{\phi}(t, t_0)$. Note that the computed $\boldsymbol{\phi}(t, t_0)$ would be different for different values of t_0 for time-varying systems.

4.2 Discrete-Time Systems

The solutions of the system equations for linear, discrete-time systems described either by Eqs. (10) and (11) or by Eqs. (12) and (13) are given in Table 5. Expressions for the state transition matrix $\boldsymbol{\phi}(k - k_0)$ for time-invariant systems and $\boldsymbol{\phi}(k, k_0)$ for the time-varying systems and the properties of the state transition matrix are also included in the table. These results can be found in standard textbooks on state-space methods.[2,4]

The state transition matrix $\boldsymbol{\phi}(k - k_0)$ for a time-invariant system depends only on the difference in the sequence number $(k - k_0)$. The transition matrix is given by

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Table 5 Solution of the State-Space Equations for Linear, Discrete-Time Systems

	Time-Invariant System [Eqs. (12) and (13)]	Time-Varying System [Eqs. (10) and (11)]
Solution $k > k_0$	$\mathbf{x}(k) = \boldsymbol{\phi}(k - k_0)\mathbf{x}(k_0) + \sum_{m=k_0}^{k-1} \boldsymbol{\phi}(k - m - 1)\mathbf{G}\mathbf{u}(m)$	$\mathbf{x}(k) = \boldsymbol{\phi}(k, k_0)\mathbf{x}(k_0) + \sum_{m=k_0}^{k-1} \boldsymbol{\phi}(k, m + 1)\mathbf{G}(m)\mathbf{u}(m)$
	$\mathbf{y}(k) = \mathbf{C}\boldsymbol{\phi}(k - k_0)\mathbf{x}(k_0) + \sum_{m=k_0}^{k-1} [\mathbf{C}\boldsymbol{\phi}(k - m - 1)\mathbf{G}\mathbf{u}(m)] + \mathbf{D}\mathbf{u}(k)$	$\mathbf{y}(k) = \mathbf{C}(k)\boldsymbol{\phi}(k, k_0)\mathbf{x}(k_0) + \sum_{m=k_0}^{k-1} [\mathbf{C}(k)\boldsymbol{\phi}(k, m + 1)\mathbf{G}(m)\mathbf{u}(m)] + \mathbf{D}(k)\mathbf{u}(k)$
State transition matrix	$\boldsymbol{\phi}(k - k_0) = \mathbf{F}^{k-k_0} \quad k > k_0$ or $\boldsymbol{\phi}(k) = \mathcal{Z}^{-1}[z(zI - F)^{-1}]$	$\boldsymbol{\phi}(k, k_0) = \begin{cases} \prod_{l=k}^{k-1} \mathbf{F}(l) & k > k_0 \\ \mathbf{I} & k = k_0 \end{cases}$
Properties of state transition matrix	$\boldsymbol{\phi}(0) = \mathbf{I}$ $\boldsymbol{\phi}(k_1 + k_2) = \boldsymbol{\phi}(k_1)\boldsymbol{\phi}(k_2)$ $\boldsymbol{\phi}(k) = \boldsymbol{\phi}^{-1}(-k)$ when the inverse exists	$\boldsymbol{\phi}(k_0, k_0) = \mathbf{I}$ $\boldsymbol{\phi}(k_2, k_1)\boldsymbol{\phi}(k_1, k_0) = \boldsymbol{\phi}(k_2, k_0)$ $\boldsymbol{\phi}(k_1, k_2) = \boldsymbol{\phi}^{-1}(k_2, k_1)$ when the inverse exists

$$\boldsymbol{\phi}(k - k_0) = \mathbf{F}^{k-k_0} \tag{50}$$

Numerical computation of $\boldsymbol{\phi}(k - k_0)$ is thus straightforward. Analytical evaluation of $\boldsymbol{\phi}(k - k_0)$ using Eq. (50) is feasible if $\mathbf{F}$ is in the diagonal or nondiagonal Jordan canonical form. If $\mathbf{F}$ is a diagonal matrix with diagonal elements equal to its eigenvalues z_i, $i = 1, \ldots, n$, then

$$\boldsymbol{\phi}(k) = \mathbf{F}^k = \begin{bmatrix} z_1^k & & & \\ & z_2^k & & \mathbf{0} \\ & & \ddots & \\ \mathbf{0} & & & z_n^k \end{bmatrix} \tag{51a}$$

If $\mathbf{F}$ is in the nondiagonal Jordan canonical form, analytical evaluation of $\boldsymbol{\phi}(k)$ is only slightly more complex. If the transformation matrix $\mathbf{T}$ [Eq. (17)] was used to obtain the diagonal or nondiagonal Jordan matrix $\mathbf{F}$, the transition matrix, for the original state vector $\mathbf{T}^{-1}\mathbf{x}$, is $\mathbf{T}^{-1}\mathbf{F}^k\mathbf{T}$.

An alternative analytical evaluation of the transition matrix uses the relationship

$$\boldsymbol{\phi}(k) = \mathcal{Z}^{-1}[z(z\mathbf{I} - \mathbf{F})^{-1}] \tag{51b}$$

where $\mathcal{Z}^{-1}$ denotes the inverse z-transform. This method is useful only for low-order systems because of the need for inverting the matrix $z\mathbf{I} - \mathbf{F}$, which has symbolic elements. The matrix inversion is particularly simple if $\mathbf{F}$ is in the diagonal or Jordan canonical form. As for continuous-time systems, the matrix inversion can be avoided altogether by using simulation diagrams of the discrete-time system and block diagram reduction[2] to directly determine elements of the matrix $(z\mathbf{I} - \mathbf{F})^{-1}$.

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The computation of the state transition matrix $\boldsymbol{\phi}(k, k_0)$ for time-varying, discrete-time systems is difficult, as it is for continuous-time systems. For small or moderate values of the order n of the system, numerical evaluation of Eq. (52) is appropriate:

$$\boldsymbol{\phi}(k, k_0) = \begin{cases} \prod_{l=k_0}^{k-1} \mathbf{F}(l) & k > k_0 \\ \mathbf{I} & k = k_0 \end{cases} \tag{52}$$

For larger values of n, analytical and numerical methods for determining $\boldsymbol{\phi}(k, k_0)$ in special cases are available.[2]

5 STABILITY

Since state-space formulation is applicable to a large class of dynamic systems, the question of stability for systems represented in state space is quite a complex one. A more general consideration of stability than that used for SISO, LTI systems would indicate that stability of dynamic systems is not really a property of the systems but is more properly associated with isolated equilibrium points of dynamic systems.[4] A particular point $\mathbf{x}_e$ in state space is an equilibrium point of a dynamic system if, in the absence of inputs, the system state $\mathbf{x}$ is equal to $\mathbf{x}_e$ for time $t \geq t_0$ for continuous-time systems or for $k \geq k_0$ for discrete-time systems. For linear systems described by the state-space equations given in Section 2, the only isolated equilibrium point is at the origin in state space. For nonlinear systems, there may be a number of isolated equilibrium points. Any isolated equilibrium point can be shifted to the origin in state space by a simple change of state variables.[4] The stability definitions to be given assume therefore that the equilibrium point is at the origin in state space and that the system is unforced. Only the more commonly used types of stability will be defined.

The origin is a stable equilibrium point if, for any given value $\epsilon > 0$, there exists a number $\kappa(\epsilon, t_0) > 0$ such that, if the norm $\|\mathbf{x}(t_0)\| < \kappa$, then the norm $\|\mathbf{x}(t)\| < \epsilon$ for all $t > t_0$. The norm of a vector $\mathbf{x}$ may be defined as the Euclidean norm:

$$\|\mathbf{x}(t)\| = \sqrt{\sum_{i=1}^{n} x_i^2(t)} \tag{53}$$

The origin is asymptotically stable if, in addition to being stable, there exists a number $\gamma(t_0) > 0$ such that whenever $\|\mathbf{x}(t_0)\| < \gamma(t_0)$ the following condition is satisfied:

$$\lim_{t \to \infty} \|\mathbf{x}(t)\| = 0 \tag{54}$$

If κ and γ are not functions of t_0 in the previous definitions, the origin is said to be uniformly stable or uniformly asymptotically stable, respectively. If $\gamma(t_0)$ can be arbitrarily large, the origin is said to be globally asymptotically stable. Extension of these stability definitions to discrete-time systems is straightforward and merely requires that the sequence numbers k, k_0 be used instead of the time instants t, t_0, respectively, in the definitions already given. Additional types of stability that depend on the inputs to the system have been defined by Brogan[4] and Kuo.[7]

For LTI systems, the conditions for stability reduce to conditions on the eigenvalues of the system matrix $\mathbf{A}$ or $\mathbf{F}$ and are summarized in Table 6. These eigenvalues are the roots of the system characteristic equation as well, as shown in Section 7. They may be computed

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Table 6 Stability Criteria for Linear, Time-Invariant Systems

	Continuous-Time System $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}$ Eigenvalues of $\mathbf{A}$ are $s_i = \alpha_{ic} \pm j\omega_{ic}$	Discrete-Time System $\mathbf{x}(k+1) = \mathbf{F}\mathbf{x}(k) + \mathbf{G}\mathbf{u}(k)$ Eigenvalues of $\mathbf{F}$ are $z_i = \beta_{ic} \pm j\omega_{ic}$
Asymptotically stable	$\alpha_{ic} < 0$ for all roots	$\lvert z_i \rvert < 1$ for all roots
Stable	$\alpha_{ic} < 0$ for all repeated roots and $\alpha_{ic} \leq 0$ for all simple roots	$\lvert z_i \rvert < 1$ for all repeated roots and $\lvert z_i \rvert \leq 1$ for all simple roots
Unstable	$\alpha_{ic} > 0$ for any simple root or $\alpha_{ic} \geq 0$ for any repeated root	$\lvert z_i \rvert > 1$ for any simple root or $\lvert z_i \rvert \geq 1$ for any repeated root

explicitly by numerical methods. Alternatively, stability criteria such as the Routh–Hurwitz criterion[5] for continuous-time systems or the Jury test[7] for discrete-time systems may be applied. The conditions for asymptotic stability of such systems can also be shown to be sufficient for other types of stability depending on the input, such as bounded-input, bounded-output stability.[4]

For continuous-time, LTV systems, the necessary and sufficient condition for the origin to be a stable equilibrium point is that there exists a number $N(t_0)$ such that the norm of the transition matrix satisfies the following condition:

$$\|\boldsymbol{\phi}(t, t_0)\| \leq N(t_0) \qquad \text{for } t \geq t_0 \tag{55}$$

If, in addition, $\|\boldsymbol{\phi}(t, t_0)\| \to 0$ as $t \to \infty$, the system is globally asymptotically stable.[4] The norm of the matrix $\boldsymbol{\phi}$ may be defined as the spectral norm:

$$\|\boldsymbol{\phi}(t, t_0)\| = \sqrt{\max_{\|x\|=1} (\mathbf{x}^{\mathrm{T}}\boldsymbol{\phi}^{\mathrm{T}}\boldsymbol{\phi}\mathbf{x})} \tag{56}$$

The corresponding stability conditions for linear, discrete-time systems are obtained simply by substituting the sequence numbers k and k_0 for time instants t and t_0, respectively, in the development. Time-varying systems that satisfy the property that the state converges exponentially with time to the zero state are said to be exponentially stable.[12] For LTI systems, of course, asymptotic stability is the same as exponential stability.

Stability considerations for nonlinear systems are more complex. For unforced second-order nonlinear systems, the phase-plane method is useful for examining the stability of equilibrium points of the system. The phase plane has the state variables as the coordinates. The state-space equations are used to derive analytical expressions for the trajectories or to draw the trajectories by graphical means. The phase portraits can then be examined to determine the equilibrium points and their stability. Application of the phase-plane method is described by DeRusso et al.[2] for continuous-time systems and by Kuo[7] for discrete-time systems.

Stability analysis of high-order nonlinear systems represented in state space can be done using the second method of Lyapunov. This is a technique requiring considerable ingenuity for effective use and provides sufficient conditions for stability rather than necessary and sufficient conditions.[8]

Lyapunov's method for nonlinear, unforced, time-invariant systems requires the definition of a scalar function of state $V(\mathbf{x})$ called the Lyapunov function. The latter may be thought

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of as a generalized energy function. The requirement on the Lyapunov function is that it be positive definite in some region about the origin in state space, the origin having been assumed to be an isolated equilibrium point here. A function $V(\mathbf{x})$, which is continuous and has continuous partial derivatives, is said to be positive (negative) definite in some region about the origin if it is zero at the origin and greater than (less than) zero everywhere else in the specified region. If the function is greater than (less than) or equal to zero everywhere in the specified region, it is said to be positive (negative) semidefinite.[4]

Consider the unforced continuous-time system represented by the state equation

$$\dot{\mathbf{x}}(t) = \mathbf{f}[\mathbf{x}(t)] \tag{57}$$

where

$$\mathbf{f}(\mathbf{0}) = \mathbf{0} \tag{58}$$

If a positive-definite function $V(\mathbf{x})$ can be determined in some region Γ about the origin such that its derivative with respect to time is negative semidefinite in Γ, then the origin is a stable equilibrium point. If dV/dt is negative definite, the origin is asymptotically stable. If the region Γ can be arbitrarily large and the conditions for asymptotic stability hold and if, in addition, $V(\mathbf{x}) \to \infty$ as $\|\mathbf{x}\| \to \infty$, the origin is a globally asymptotically stable equilibrium point. Table 7 gives the corresponding stability conditions for nonlinear, time-invariant, discrete-time systems. Extensions of the stability conditions for time-varying systems have been described by Kalman and Bertram[8] and DeRusso et al.[2]

As an example of the application of the second method of Lyapunov, consider the following nonlinear system:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a_0 x_2 - b_0 x_2^3 - x_1 \end{aligned} \tag{59}$$

where $a_0, b_0 \geq 0$ and both are not zero. The origin is an equilibrium point for this system since, if both x_1 and x_2 are zero,

$$\dot{x}_1 = \dot{x}_2 = 0 \tag{60}$$

Consider the following Lyapunov function:

$$V(x_1, x_2) = x_1^2 + x_2^2 \tag{61}$$

It satisfies the conditions for positive definiteness in an arbitrarily large region about the origin:

Table 7 Application of the Second Method of Lyapunov to Nonlinear, Time-Invariant, Discrete-Time Systems

State equation	$\mathbf{x}(k+1) = \mathbf{f}[\mathbf{x}(k)]$ $\mathbf{f}(\mathbf{0}) = \mathbf{0}$
Lyapunov function	Scalar function $V[\mathbf{x}(k)]$ positive definite in some region about the origin
Condition for stability in Γ	$\Delta V = V[\mathbf{x}(k+1)] - V[\mathbf{x}(k)]$ is negative semidefinite in Γ
Condition for asymptotic stability in Γ	$\Delta V = V[\mathbf{x}(k+1)] - V[\mathbf{x}(k)]$ is negative definite in Γ
Condition for global asymptotic stability	(i) Γ can be arbitrarily large (ii) $V[\mathbf{x}(k)] \to \infty$ as $\|\mathbf{x}(k)\| \to \infty$

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$$\frac{dV(x_1, x_2)}{dt} = 2x_1\dot{x}_1 + 2x_2\dot{x}_2$$

$$= -2(a_0x_2^2 + b_0x_2^4) \tag{62}$$

after using the state equations to substitute for $\dot{x}_1$ and $\dot{x}_2$. If a_0, b_0 satisfy the inequalities stated, dV/dt is negative semidefinite in an arbitrarily large region about the origin. The origin is thus a stable equilibrium point. In fact, using a corollary to the main stability theorem provided by Kalman and Bertram,[8] it can be shown that the origin is a global asymptotically stable equilibrium point.

The limitations of Lyapunov' s method are that the Lyapunov function is not unique for a system and there are no systematic procedures for finding a suitable Lyapunov function. Since only sufficient conditions for stability are determined, some choices of Lyapunov functions are better in that they provide more information about system stability than others. Also, appropriate choice of the Lyapunov function can lead to an estimate of the system speed of response.[8] In practice, therefore, the second method of Lyapunov is used primarily to analyze the stability of systems such as high-order, nonlinear systems for which other methods of stability analysis are not available.

6 CONTROLLABILITY AND OBSERVABILITY

The controllability of a linear system is a measure of the coupling between the inputs to the system and the system state. The concept of state controllability was introduced by Kalman[11] in order to clarify conditions for the existence of solutions to specific control problems.

A linear, continuous-time system is said to be state controllable at time t_0 if there exists a finite time $t_1 > t_0$ and a control function $\mathbf{u}(t)$, $t_0 < t < t_1$, that can drive the system state from any initial value to any final value at $t = t_1$. If the system is controllable for all times t_0, the system is completely state controllable.[4] A linear, discrete-time system is said to be state controllable and completely state controllable, respectively, if the sequence numbers k, k_0, k_1 are substituted for the times t, t_0, t_1, respectively, in the two previously given definitions. An additional form of controllability for continuous-time and discrete-time LTV systems is that of uniformly complete state controllability. The mathematical definition of this form of controllability may be found in Kalman.[11] This property implies that the control effort and time interval required to drive the system state to the final value is relatively independent of the initial time. For LTI systems, of course, complete state controllability is the same as uniformly complete state controllability.

Though the control problems formulated above are open-loop control problems, the property of controllability has very significant implications for closed-loop control problems. Section 2 in Chapter 18 indicates that the closed-loop poles of a completely state-controllable time-invariant system can be specified and placed arbitrarily in the complex s-plane (or z-plane for discrete-time systems) by proportional state-variable feedback. Moreover, satisfaction of the controllability conditions to be defined in this section for time-invariant systems ensures that the optimal-control law for a quadratic performance index is a proportional state-variable feedback law and yields an asymptotically stable closed-loop system.[10]

Direct application of the definition of state controllability to LTI systems yields controllability conditions involving the transition matrices. Simple algebraic conditions are usually available for such systems and are used more often in practice to evaluate controllability.

The controllability condition for LTI systems with distinct eigenvalues may be stated very simply if the state equations are transformed to the diagonal Jordan canonical form. Such systems are completely controllable if there are no zero rows in the transformed **B**

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matrix for continuous-time systems or in the transformed $\mathbf{G}$ matrix for discrete-time systems.[4] The presence of a zero row in either of these matrices would indicate that the inputs are not coupled to and cannot control the corresponding mode. Algebraic controllability conditions for systems with repeated eigenvalues are given by Palm.[6]

The controllability conditions for LTI systems in general are stated in terms of the matrix $\mathbf{P}_c$, referred to as the controllability matrix in Section 3, and are summarized in Table 8. The $n \times nr$ controllability matrix for a MIMO system is defined by

$$\mathbf{P}_c = [\mathbf{B} \quad \mathbf{AB} \quad \cdots \quad \mathbf{A}^{n-1}\mathbf{B}] \tag{63}$$

for continuous-time systems and by

$$\mathbf{P}_c = [\mathbf{G} \quad \mathbf{FG} \quad \cdots \quad \mathbf{F}^{n-1}\mathbf{G}] \tag{64}$$

for discrete-time systems. The condition for complete state controllability is simply that the matrix $\mathbf{P}_c$ has rank n. The controllable canonical form for SISO systems, described in Section 3, derives its name from the fact that transformation to that form is possible if and only if

Table 8 Controllability Conditions for Linear Dynamic Systems

	Continuous Time	Discrete Time
Time-Invariant System		
Necessary and efficient condition for state controllability	(i) $\text{rank}(\mathbf{B} \quad \mathbf{AB} \quad \cdots \quad \mathbf{A}^{n-1}\mathbf{B}) = \text{rank}(\mathbf{B} \quad \mathbf{AB} \quad \cdots \quad \mathbf{A}^{n-r}\mathbf{B}) = n$ or (ii) $\det(\mathbf{P}_c\mathbf{P}_c^{\mathrm{T}}) \neq 0$	(i) $\text{rank}(\mathbf{G} \quad \mathbf{FG} \quad \cdots \quad \mathbf{F}^{n-1}\mathbf{G}) = \text{rank}(\mathbf{G} \quad \mathbf{FG} \quad \cdots \quad \mathbf{F}^{n-r}\mathbf{G}) = n$ or (ii) $\det(\mathbf{P}_c\mathbf{P}_c^{\mathrm{T}}) \neq 0$
Necessary condition for state controllability	$\text{rank}(\mathbf{B} \; \mathbf{A}) = n$	$\text{rank}(\mathbf{G} \; \mathbf{F}) = n$
Necessary and sufficient condition for output controllability	(i) $\text{rank}(\mathbf{CP}_c) = p$ or (ii) $\det(\mathbf{CP}_c\mathbf{P}_c^{\mathrm{T}}\mathbf{C}^{\mathrm{T}}) \neq 0$	(i) $\text{rank}(\mathbf{CP}_c) = p$ or (ii) $\det(\mathbf{CP}_c\mathbf{P}_c^{\mathrm{T}}\mathbf{C}^{\mathrm{T}}) \neq 0$
Time-Varying System: Time Interval of Interest $[t_0, t_1]$ or $[k_0, k_1]$		
Necessary and sufficient condition for state controllability	(i) $\mathbf{W}_c(t_1, t_0)$ is positive definite or (ii) Zero is not an eigenvalue of $\mathbf{W}_c(t_1, t_0)$ or (iii) $\lvert\mathbf{W}_c(t_1, t_0)\rvert \neq 0$ where $\mathbf{W}_c(t_1, t_0)$ $\triangleq \int_{t_0}^{t_1} \boldsymbol{\phi}(t_1, \tau)\mathbf{B}(\tau)\mathbf{B}^{\mathrm{T}}(\tau) \times \boldsymbol{\phi}^{\mathrm{T}}(t_1, \tau)\, d\tau$	(i) $\mathbf{W}_c(k_1, k_0)$ is positive definite or (ii) Zero is not an eigenvalue of $\mathbf{W}_c(k_1, k_0)$ or (iii) $\lvert\mathbf{W}_c(k_1, k_0)\rvert \neq 0$ where $\mathbf{W}_c(k_1, k_0)$ $\triangleq \sum_{k=k_0}^{k_1} \boldsymbol{\phi}(k_1, k)\mathbf{G}(k)\mathbf{G}^{T}(k) \times \boldsymbol{\phi}^{\mathrm{T}}(k_1, k)$
Necessary and sufficient condition for output controllability	$\det \mathbf{W}_y(t_1, t_0) \neq 0$ where $\mathbf{W}_y(t_1, t_0)$ $\triangleq \int_{t_0}^{t_1} \mathbf{C}(\tau)\boldsymbol{\phi}(t_1, \tau)\mathbf{B}(\tau)\mathbf{B}^{\mathrm{T}}(\tau) \times \boldsymbol{\phi}^{\mathrm{T}}(t_1, \tau)\mathbf{C}^{\mathrm{T}}(\tau)\, d\tau$	$\det \mathbf{W}_y(k_1, k_0) \neq 0$ where $\mathbf{W}_y(k_1, k_0)$ $\triangleq \sum_{k=k_0}^{k_1} \mathbf{C}(k)\boldsymbol{\phi}(k_1, k)\mathbf{B}(k) \times \mathbf{B}^{\mathrm{T}}(k)\boldsymbol{\phi}^{\mathrm{T}}(k_1, k)\mathbf{C}^{\mathrm{T}}(k)$

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the system is completely state controllable. The transformation matrix $\mathbf{T}$ in Tables 2 and 4 required to transform the state-space equations for a SISO system to the controllable canonical form exists if and only if the $n \times n$ matrix $\mathbf{P}_c$ is nonsingular; that is, the system is controllable. Equivalent controllability conditions that are simpler to evaluate than the one previously stated are also listed in Table 8 along with a necessary (but not sufficient) condition for complete controllability.

The controllability conditions for LTV systems over a specified time interval are more cumbersome to evaluate in practice as they involve the system transition matrix.[4] These conditions are listed in Table 8. In contrast to time-invariant systems, the controllability of time-varying systems depends on the time interval under consideration.

The concept of output controllability, as opposed to state controllability described earlier, was introduced by Kreindler and Sarachik.[13] A linear, continuous-time system is said to be output controllable at time t_0 if there exists a finite time $t_1 > t_0$ and a control function $\mathbf{u}(t)$, $t_0 < t < t_1$, that drives the system output from any initial value $\mathbf{y}(t_0)$ to any final value $y(t_1)$. If this condition holds true for all times t_0, the system is completely output controllable. Extension of the concept to linear, discrete-time systems is straightforward as before.

Output controllability conditions[4] for linear systems that are purely dynamic [i.e., the matrix $\mathbf{D} = \mathbf{0}$ in Eqs. (5), (7), (11), and (13)] are summarized in Table 8. These conditions are weaker than the corresponding conditions for state controllability if the number of outputs p is less than the number of state variables n. Since this is true in practice, state controllability implies output controllability. On the other hand, output controllability does not imply state controllability in general. It can be shown, however, that for time-invariant systems if the matrix $(\mathbf{CC}^{\mathrm{T}})$ is nonsingular, output controllability is equivalent to state controllability.

The observability of a linear system is a measure of the coupling between the system state and its outputs. The concept of observability was introduced by Kalman[11] and is relevant to the problem of estimation of system state based on the output vector. The output vector is usually chosen to correspond to measurable variables.

A linear, continuous-time system is said to be observable[4] at time t_0 if there exists a finite time $t_1 > t_0$ such that $\mathbf{x}(t_0)$ can be determined from the history of inputs $\mathbf{u}(t)$ and outputs $\mathbf{y}(t)$ over the time interval $t_0 \leq t \leq t_1$. If the system is observable for all times t_0 and all initial states $\mathbf{x}(t_0)$, the system is completely observable. Extension of the observability concept to discrete-time systems simply requires that the sequence numbers k, k_0, k_1 be substituted for the times t, t_0, t_1, respectively, in the previous definitions. A stronger form of observability for LTV systems is that of uniformly complete observability. The mathematical definition of this form of observability is given by Kalman.[11] This property guarantees that the time interval required to estimate the state is relatively independent of the initial time. For LTI systems, of course, complete observability is the same as uniformly complete observability. A property complementary to observability for LTV systems is that of reconstructibility,[12,14] which concerns the estimation of the state of the system from past measurements of the state. In contrast to this, observability concerns the estimation of the state from future measurements of the output. For time-invariant systems, the two properties of reconstructibility and observability are identical to one another.

As was the case for controllability, direct application of the definition of observability already stated yields conditions involving the transition matrix.[4] Simpler algebraic conditions are available for time-invariant systems. The observability condition for LTI systems with distinct eigenvalues can be stated very simply if the state equations are transformed to the Jordan canonical form. Such systems are completely observable if each column in the transformed $\mathbf{C}$ matrix has at least one nonzero element.[4] The presence of a column of zeros in this matrix would indicate that the corresponding state variable cannot be estimated from the measured output and input vectors.

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More general observability conditions for LTI systems are stated in terms of a matrix $\mathbf{P}_0$ referred to as the observability matrix and are summarized in Table 9. The $np \times n$ observability matrix is defined by

$$\mathbf{P}_0 = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix} \tag{65}$$

for continuous-time systems and by

$$\mathbf{P}_0 = \begin{bmatrix} \mathbf{C} \\ \mathbf{CF} \\ \vdots \\ \mathbf{CF}^{n-1} \end{bmatrix} \tag{66}$$

for discrete-time systems. The condition for complete observability is simply that the matrix $\mathbf{P}_0$ have rank n. The observable canonical form for SISO systems, described in Section 3, derives its name from the fact that transformation to that form is possible if and only if the system is observable. The transformation matrix $\mathbf{T}$ in Tables 2 and 4, required to transform the state-space equations for a SISO system to the observable canonical form, exists if and only if the $n \times n$ matrix $\mathbf{P}_0$ is nonsingular, that is, the system is observable. Equivalent observability conditions which are simpler to evaluate than the one stated previously are also listed in Table 9 along with a necessary (but not sufficient) condition for complete observability. It should be noted that the observability conditions are independent of time for time-

Table 9 Observability Conditions for Linear Dynamic Systems

	Continuous Time	Discrete Time
Time-Invariant System		
Necessary and sufficient condition for observability	(i) $\text{rank}[\mathbf{C}^T \quad \mathbf{A}^T\mathbf{C}^T \quad \cdots \quad (\mathbf{A}^T)^{n-1}\mathbf{C}^T] = \text{rank}[\mathbf{C}^T \quad \mathbf{A}^T\mathbf{C}^T \quad \cdots \quad (\mathbf{A}^T)^{n-p}\mathbf{C}^T] = n$ or (ii) $\det(\mathbf{P}_0^T\mathbf{P}_0) \neq 0$	(i) $\text{rank}[\mathbf{C}^T \quad \mathbf{F}^T\mathbf{C}^T \quad \cdots \quad (\mathbf{F}^T)^{n-1}\mathbf{C}^T] = \text{rank}[\mathbf{C}^T \quad \mathbf{F}^T\mathbf{C}^T \quad \cdots \quad (\mathbf{F}^T)^{n-p}\mathbf{C}^T] = n$ or (ii) $\det(\mathbf{P}_0^T\mathbf{P}_0) \neq 0$
Necessary condition	$\text{rank}(\mathbf{C}^T \; \mathbf{A}^T) = n$	$\text{rank}(\mathbf{C}^T \; \mathbf{F}^T) = n$
Time-Varying Systems:		
Necessary and sufficient condition for observability	Observable at t_0 if and only if there exists a finite time t_1, $t_1 > t_0$ such that (i) $\mathbf{W}_0(t_1, t_0)$ is positive definite or (ii) zero is not an eigenvalue of $\mathbf{W}_0(t_1, t_0)$ or (iii) $\lvert\mathbf{W}_0(t_1, t_0)\rvert \neq 0$ where $\mathbf{W}_0(t_1, t_0) \triangleq \int_{t_0}^{t_1} \boldsymbol{\phi}^T(\tau, t_0)\mathbf{C}^T(\tau)\mathbf{C}(\tau) \times \boldsymbol{\phi}(\tau, t_0)\, d\tau$	Observable at k_0 if and only if there exists a finite time k_1, $k_1 > k_0$ such that (i) $\mathbf{W}_0(k_1, k_0)$ is positive definite or (ii) zero is not an eigenvalue of $\mathbf{W}_0(k_1, k_0)$ or (iii) $\lvert\mathbf{W}_0(k_1, k_0)\rvert \neq 0$ where $\mathbf{W}_0(k_1, k_0) \triangleq \sum_{k=k_0}^{k_1} \boldsymbol{\phi}^T(k_1, k_0)\mathbf{C}^T(k)\mathbf{C}(k)\boldsymbol{\phi}(k_1, k)$

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invariant systems. In contrast, the observability conditions for time-varying systems over a specified time interval involve the system transition matrix and hence depend on the time interval.[4] They are also listed in Table 9.

Though the definition of observability above involves an open-loop state estimation problem, the property of observability has important implications for closed-loop realizations of the state estimation problem. It will be shown in Section 5 of Chapter 18 that, if a time-invariant system is completely observable, a closed-loop state estimator can be constructed such that the estimation error transients can be made to decay to zero as rapidly as possible.

The conditions for controllability and observability noted in Tables 8 and 9 have obvious similarities. The two properties can be shown to be duals of each other by formulating the concept of the dual of a dynamic system. Interested readers are referred to Kalman et al.[11,14]

A linear system can, in general, be divided into four subsystems as indicated by Fig. 4. The state vector $\mathbf{x}$ can be written as

$$x^{\mathrm{T}} = x_C^{\mathrm{T}} + x_{CO}^{\mathrm{T}} + x_N^{\mathrm{T}} + x_O^{\mathrm{T}} \tag{67}$$

where the subscripts have the meaning assigned in the figure. The corresponding state-space equations for a time-invariant, continuous-time system are

$$\begin{bmatrix} \dot{\mathbf{x}}_C \\ \dot{\mathbf{x}}_{CO} \\ \dot{\mathbf{x}}_N \\ \dot{\mathbf{x}}_O \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \mathbf{A}_{13} & \mathbf{A}_{14} \\ \mathbf{0} & \mathbf{A}_{22} & \mathbf{0} & \mathbf{A}_{24} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_{33} & \mathbf{A}_{34} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{44} \end{bmatrix} \begin{bmatrix} \mathbf{x}_C \\ \mathbf{x}_{CO} \\ \mathbf{x}_N \\ \mathbf{x}_O \end{bmatrix} + \begin{bmatrix} \mathbf{B}_{11} \\ \mathbf{B}_{21} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \mathbf{u} \tag{68}$$

$$\mathbf{y} = [\mathbf{0} \quad \mathbf{C}_{12} \quad \mathbf{0} \quad \mathbf{C}_{14}] \begin{bmatrix} \mathbf{x}_C \\ \mathbf{x}_{CO} \\ \mathbf{x}_N \\ \mathbf{x}_O \end{bmatrix} + \mathbf{D}\mathbf{u}$$

The zero matrices in the $\mathbf{B}$ matrix correspond to the fact that $\mathbf{x}_N$ and $\mathbf{x}_O$ are not controllable. The zero matrices in the $\mathbf{C}$ matrix correspond to the fact that $\mathbf{x}_C$ and $\mathbf{x}_N$ are not observable. If the eigenvalues of the $\mathbf{A}$ matrix are distinct, all off-diagonal elements in the $\mathbf{A}$ matrix would be zero. The procedure for determining the transformation matrix to convert the state-space equations into the canonical form [Eq. (68)] has been described by Kalman.[9]

The extension of this discussion to time-invariant, discrete-time systems is straightforward. For time-varying systems, the state-space decomposition is a function of time but is similar in structure to that already described.

The significance of the system decomposition as shown in Fig. 4 is that it helps relate the state-space description of linear dynamic systems to transfer function or transfer function matrix descriptions of such systems. The transfer function matrix relating $\mathbf{y}$ to $\mathbf{u}$ is a description only of the controllable and observable part of the system and masks other modes that are either not observable or not controllable or neither controllable nor observable. The relationship of the state-space description of dynamic systems to the transfer function matrix description of such systems is discussed in greater detail in Section 7.

Loss of controllability or observability could occur when controllable and observable subsystems are connected together to form composite systems. Gilbert[15] has formulated rules relating the composite system properties to those of the individual open-loop systems. These rules provide greater insight into the conditions leading to loss of controllability or observability than the simple application of the conditions noted in Tables 8 and 9.

The concepts of controllability and observability are obviously very important for MIMO systems since the complexity of such systems frequently masks the nature of the coupling of the system state to the inputs and outputs. For SISO systems, lack of complete

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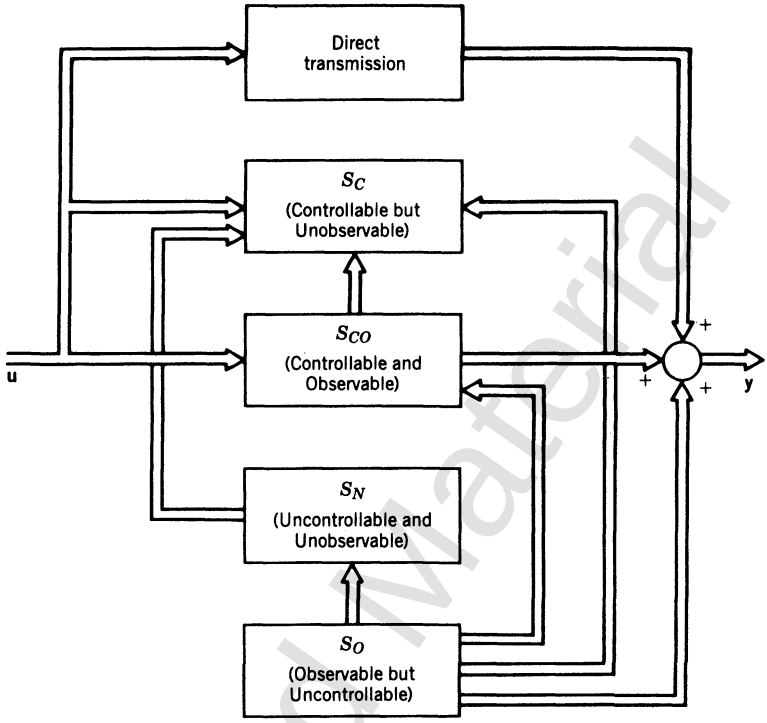


Figure 4 Decomposition of linear system based on controllability and observability.

controllability or observability is a less common occurrence. Conclusions concerning controllability and observability are obvious in many of these cases but less so in others.

Consider the electrical circuit in Fig. 5. The state-space equation for the system is given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\dfrac{R_1 + R_3}{L_1} & \dfrac{R_3}{L_1} \\ \dfrac{R_3}{L_2} & -\dfrac{R_3 + R_2}{L_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \dfrac{1}{L_1} \\ 0 \end{bmatrix} u(t) \tag{69}$$

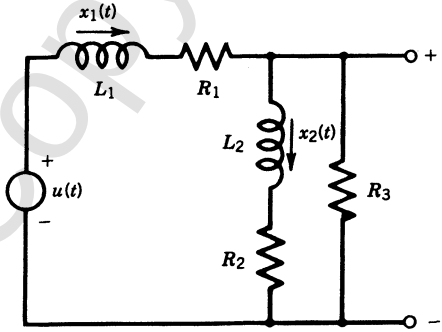


Figure 5 State variables for *RL* circuit.

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The system is completely controllable except for the trivial case where R_3 is zero. Similarly, if x_1 or x_2 is chosen as the only output, the system is completely observable as long as R_3 is nonzero. If the voltage across the resistor R_3 is chosen as the output, the corresponding output equation is

$$y(t) = [R_3 \; -R_3] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{70}$$

The observability matrix $\mathbf{P}_0$ in Eq. (65) is then nonsingular and the system is observable if and only if

$$\frac{R_1}{L_1} \neq \frac{R_2}{L_2} \tag{71}$$

This observability condition is not an obvious one and is equivalent to the requirement that the time constants associated with the two R–L pairs not be identical. However, it should be noted that, given component tolerances in practice, the inequality (71) will be satisfied almost always and the corresponding system will be observable. A discussion on conditions leading to loss of controllability or observability is given by Friedland.[16]

Despite the fact that lack of complete controllability or observability is infrequent for SISO systems, these concepts have practical significance for SISO as well as MIMO systems because of the relationship of these concepts to closed-loop control and state estimation problems. Measures of the degree of controllability and observability can be defined for time-invariant and time-varying systems.

The controllability and observability conditions in Tables 8 and 9 relate these properties to the nonsingularity of square matrices for SISO systems. Measures of the degree of controllability and observability are related to the closeness of these matrices to the singularity condition. Such measures have been defined for time-invariant systems by Johnson[17] and Friedland[18] and are significant for SISO as well as MIMO systems. A system with a better degree of controllability can in general be controlled more effectively. Similarly, a better degree of observability implies that state estimation can be performed more accurately. The proposed measures of the degree of controllability and observability are not in common use but have the potential to quantitatively evaluate proposed control strategies and measurement schemes.[19]

Additional concepts of degrees of controllability and observability for time-varying systems have been described by Silverman and Meadows[20] and for MIMO systems by Kreindler and Sarachik.[13] Properties weaker than state controllability and observability have also been defined[12] and are useful in ensuring that closed-loop control and state estimation problems are well posed. A linear system is said to be stabilizable if the uncontrollable subsystems S_N and S_O in the decomposition of Fig. 4 are stable. Similarly, if the unobservable subsystems S_C and S_N are stable, the system is said to be detectable.[12]

7 RELATIONSHIP BETWEEN STATE-SPACE AND TRANSFER FUNCTION DESCRIPTIONS

The state-space representation of dynamic systems is an accurate representation of the internal structure of a system and its coupling to the system inputs and outputs. For LTI systems, transfer functions (for SISO systems) or transfer function matrices (for MIMO systems) are useful in practice since the dimensions of these matrices are invariably smaller than the dimensions of the corresponding system matrices $\mathbf{A}$ or $\mathbf{F}$ in Eqs. (6) and (12). Analysis and design procedures based on the transfer function matrix descriptions are there-

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fore simpler. The relationship between these two alternative descriptions of LTI systems is described in this section.

Determination of the transfer function matrix from the state-space equations is straightforward. For continuous-time systems, Laplace transformation of Eqs. (6) and (7) with zero initial conditions $\mathbf{x}(0)$ and elimination of $\mathbf{X}(s)$ yields

$$\mathbf{H}(s) = \frac{\mathbf{Y}(s)}{\mathbf{U}(s)} = \mathbf{C}(s\mathbf{I}_n - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} \tag{72}$$

where $\mathbf{I}_n$ is the $n \times n$ identity matrix. For discrete-time systems, a similar procedure using z-transforms and applied to Eqs. (12) and (13) yields the pulse transfer function matrix

$$\mathbf{H}(z) = \frac{\mathbf{Y}(z)}{\mathbf{U}(z)} = \mathbf{C}(z\mathbf{I}_n - \mathbf{F})^{-1}\mathbf{G} + \mathbf{D} \tag{73}$$

The transfer function matrix corresponding to a given state-space description is therefore unique. However, as indicated in the previous section, the former represents only the controllable and observable part of a system. Unless the entire system is completely controllable and observable, a transfer function matrix description is not a complete characterization of the system dynamic behavior. It can be shown that, for SISO systems, a necessary and sufficient condition for controllability and observability of the system is that there are no pole–zero cancellations between the numerator and denominator of the transfer function matrices in Eqs. (72) and (73). For MIMO systems, this is only a sufficient condition and not a necessary one.[4]

The determination of the state-space description corresponding to a given transfer function matrix description is more complex and is referred to as the problem of realization. Since the transfer function matrix represents only the controllable and observable part of a system, the problem of realization does not have a unique solution. In fact, the transfer function matrix description does not even determine the dimension of the corresponding state vector uniquely. The minimal dimension of the state vector corresponding to a given transfer function matrix is, however, uniquely determined. The associated state-space equations are said to constitute the minimal or irreducible realization of the transfer function matrix. It can be shown that a realization is minimal if and only if it is both controllable and observable.[4] Minimal realizations are not unique. However, any two different minimal realizations of a given transfer function matrix are equivalent in that the corresponding state vectors are related by a nonsingular transformation matrix.

The canonical forms of the state-space equations for SISO systems in Tables 1 and 3 represent minimal realizations if there is no pole–zero cancellation. Techniques for obtaining minimal realizations for MIMO systems are more involved. Brogan[4] has described a procedure for obtaining a Jordan form realization for a given transfer function matrix. When applied to cases where elements of the transfer function matrix have only simple poles, the resulting realization is controllable and observable, as shown in the following example. If one or more elements of the transfer function matrix have repeated poles, the realization that results is controllable but may or may not be observable.

Consider a continuous-time system with two inputs and two outputs and the following transfer function matrix:

$$\mathbf{H}(s) = \begin{bmatrix} \dfrac{1}{s+1} & \dfrac{s}{(s+1)(s+3)} \\ \dfrac{1}{s+3} & \dfrac{1}{s+1} \end{bmatrix} \tag{74}$$

Expand $\mathbf{H}(s)$ using a matrix version of partial-fraction expansion as

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$$\mathbf{H}(s) = \frac{\begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{bmatrix}}{s+1} + \frac{\begin{bmatrix} 0 & \frac{3}{2} \\ 1 & 0 \end{bmatrix}}{s+3}$$

$$= \frac{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & -\frac{1}{2} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix}}{s+1} + \frac{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 0 & \frac{3}{2} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix}}{s+3} \tag{75}$$

It should be noted that the number of vector products each coefficient matrix is factored into is equal to the rank of the matrix. Then $\mathbf{H}(s)$ is written in a form that indicates the matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ clearly, by comparison with $\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$:

$$\mathbf{H}(s) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{s+1} & & \mathbf{0} & \\ & \frac{1}{s+1} & & \\ & \mathbf{0} & \frac{1}{s+3} & \\ & & & \frac{1}{s+3} \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \\ 0 & \frac{3}{2} \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} s+1 & & & \\ & s+1 & \mathbf{0} & \\ \mathbf{0} & & s+3 & \\ & & & s+3 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \\ 0 & \frac{3}{2} \\ 1 & 0 \end{bmatrix} \tag{76}$$

Thus, the corresponding realization is

$$\mathbf{A} = \begin{bmatrix} -1 & & & \\ & -1 & \mathbf{0} & \\ & \mathbf{0} & -3 & \\ & & & -3 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \\ 0 & \frac{3}{2} \\ 1 & 0 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \tag{77}$$

The realization is controllable and observable and hence minimal. Modifications of this procedure for cases where $\mathbf{H}(s)$ has elements with repeated poles are described by Brogan.[4] Extensions to discrete-time systems are straightforward.

An alternative two-step procedure for determining a minimal realization for a transfer function matrix involves obtaining a nonminimal realization by any one method as the first step. For example, one of the many realizations in Table 1 (Table 3 for discrete-time systems) can be chosen to represent each of the elements of the transfer function matrix. The state-space descriptions of the elements can then be combined to get the state-space equations for the MIMO system. The resulting realization would, in general, be nonminimal. The second step requires transformation of the state-space equations to the form given by Eq. (68) or an equivalent one for discrete-time systems. Techniques for selecting the transformation matrix are described by Kalman[9] and Fortmann and Hitz.[1] The minimal realization is then given by the controllable and observable subsystem in Fig. 4. The resulting equations for a continuous-time system are

$$\dot{\mathbf{x}}_m = \mathbf{A}_{22}\mathbf{x}_m + \mathbf{B}_{21}\mathbf{u} \tag{78}$$

$$\mathbf{y}_m = \mathbf{C}_{12}\mathbf{x}_m + \mathbf{D}\mathbf{u} \tag{79}$$

where the subscript m indicates a minimal realization. Similar results for discrete-time systems are given by Brogan,[4] Kuo,[7] and Kalman.[9]

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8 CONCLUSION

The state-space methods presented in this chapter offer a unifying framework for the dynamic analysis and control of a variety of systems. The primary emphasis in these methods on linear time-invariant systems is a reflection of the state of the literature on the subject and the practice of the art. Results for linear time-varying systems have been given in some of the standard texts[2–4] referred to. The application of state-space methods to nonlinear system analysis and control is treated at some length by Hedrick and Paynter.[21]

Distributed-parameter systems are examples of systems with infinite-dimensional states. Application of state-space methods to these systems has been described by Tzafestas et al.[22] Time-delayed systems are also examples of systems with infinite-dimensional states. The analysis and control of such systems and of many of the other types of systems referred to in this section remains a subject of current research. For current research results in these areas, the reader is referred to journals such as the *ASME Journal of Dynamic Systems, Measurements and Controls; IEEE Transactions on Automatic Control; AIAA Journal of Guidance, Control and Dynamics; SIAM Journal on Control; and Automatica, the Journal of the International Federation of Automatic Control.*

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